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VOLUME XXII OCTOBER, 1929 NUMBER 6

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THE MATHEMATICS TEACHER

VOLUME XXII OCTOBER, 1929 NUMBER 6

ADJUSTING ALGEBRA TO ABILITY LEVELS THROUGH THE TIME ELEMENT [1]

BY JESSIE M. CLINE

High School, Bloomington, Illinois

I have been asked to tell you what we are doing in the Bloomington High School to adjust algebra to the different ability levels of our pupils; what problems we have encountered in making such adjustment; and what results we have accomplished.

I want to make it clear from the outset that our plan is in the experimental stage; and, therefore, that the features which may ultimately characterize it, or even its continuance will depend on the results achieved through it.

A word of explanation regarding our mathematics program is necessary in understanding the development of this plan. For some years we have required one and a half units of algebra in the freshman and sophomore years. Recently, we added a semester of algebra as an elective for juniors and seniors. No one has been permitted to enroll in this advanced course who had made less than a C-average in the previous courses, and no one has been allowed to continue it who failed to maintain that average. In other words, this course was created for those students who had made good in mathematics, who enjoyed it, and who expected to continue the study of it beyond high school.

The advanced course has seemed satisfactory, but it is too new for us to know exactly what we have accomplished by it. Reports from the universities as to the preparation of students who have gone from this course to college algebra, are looked forward to as a means of determining to what extent it is serving the purpose for which it is intended.

[1] Read at the State High School Conference in Urbana, Illinois, November 24, 1928.

It was in the required courses that we faced our problems. Increased school attendance during recent years brought us individuals of lower ability levels than we had considered in the making of our program. This decreasing fitness of pupils caused us to be confronted with, at least, three undesirable situations: the subject-matter for a given semester could not be completed; the percentage of failures was appalling; and good students were not stimulated to their best work.

Such conditions demanded most earnest consideration and challenged us to seek a remedy. We changed our text book in the hope of improving the situation, but results, in these respects, were much the same as before.

Two possibilities remained. Either we had to change our curriculum so that little or no algebra was necessary for graduation, or we had to adjust the courses we had been requiring so that better results could be accomplished. After much thought and deliberation, we decided to adopt the latter policy.

Our study of weak pupils has led us to the conclusion that usually their difficulty was not lack of ability to learn, but inability to work at the normal rate of speed. In accordance with this theory, we planned to create homogeneous groups and to permit each to progress at the rate determined by the ability of the pupils composing it.

The first step in inaugurating such a plan was to determine the personnel of the sections. Since we had already observed in studying our failures that a large percentage of those who could not get algebra did not know arithmetic, a test in arithmetic was decided upon as the chief determining factor.

Accordingly during the third week of the second semester last year all 9B algebra classes were given an arithmetic test. It was not a standardized test but one devised by teachers in the department. It consisted of twenty-five exercises in the fundamental operations of common fractions, decimals, and mixed numbers, along with a few thought problems. It was given by the regular teacher of each class and the papers were graded by her. The result of this test, together with the teacher's judgment, determined the placement of the pupil in a normal or slow section in algebra.

The next step in the working out of this new plan was to determine the method of procedure in each kind of group.

It was agreed that all sections were to use the same text book and the exercise book which supplements it.

The normal sections were expected to progress at the rate of speed suggested in our outline of subject-matter and to do as much supplementary work as the maintenance of this schedule permitted. If pupils were discovered in a normal group who could not work at the rate prescribed, they were shifted to slow sections.

The policy to be followed in each slow section was left to its particular teacher. She was to study the individuals of her class and to decide in the light of such study what topics in algebra were to be taken up; how much time should be given each of these topics; and how difficult the tasks within them should be.

The placement test had pointed out that these pupils were lacking in arithmetic abilities. Since the greatest reach of my imagination could not conceive of teaching algebra—generalized number—to pupils who knew almost nothing of particular numbers, I decided my section must start by building a foundation in arithmetic so that the fundamental processes would be automatic. The problem was how to teach arithmetic to pupils, who felt that they had just completed it and had left it behind them, without killing interest in the course.

As I studied the situation, I realized that it was a phase of a big problem of attitude consistent with learning; for here was a group of pupils all of whom had been selected because of deficiencies in mathematics and, whether their inabilities were native or due merely to poor training, the pupils felt branded by them. They rated themselves as failures in this particular line and consequently could not enter upon the course with anything but misgivings. It would be the teacher's task to develop an atmosphere of hope which would ripen into confidence on the basis of work well done.

Some of the devices used to accomplish this purpose were as follows: Exposing the mistakes and difficulties of an individual pupil to other members of the class was minimized by working at the seats, rather than at the board, and by not calling upon a pupil to explain his solution of a problem until the teacher had assured herself that it was right. Volunteer recitations were accepted more often than is usually deemed advisable. Efforts were made to arouse a spirit of play in work and to develop an

attitude of wholesome rivalry. No opportunity was overlooked to commend good work, while criticisms which might embarrass or discourage was resorted to as sparingly as possible. Thus, the removing of the dread of having inabilities disclosed, the infusing of a spirit of fun and competition into the work, and the use of praise, begot confidence and furnished motivating force.

Working under these incentives, the class spent four weeks during which a part of each recitation was used in learning the fundamental processes of arithmetic while all the homework was practice in developing skill in these processes, without any apparent lessening of interest. At the end of this time, the same test in arithmetic which had been given earlier in the semester was repeated and we found that we had raised the class median from 40 to 80.

The work in algebra itself started with representing ideas by algebraic symbols and the reverse process of translating algebraic expressions into words, in order that the pupils might see at the outset that algebraic expressions are not mere arbitrary combinations of symbols, but that they have definite meanings. Then followed, little by little, the evaluation and derivation of easy formulas, the solution of simple equations, addition, subtraction, multiplication, and division of signed numbers, and the interpretation and making of graphs.

One phase of the work which was particularly stressed was that which would develop ability to use mathematical language in the expression of simple quantitative relations. These pupils were handicapped in their power of obtaining information from the printed page and in their ability to express themselves because of their ignorance of such words as sum, difference, quotient, product, divisor, and dividend. There was no attempt made to learn formal definitions, but such terms were made a part of their spoken and written vocabularies by constant use in the description of processes and through sentence completion in their exercise books.

Thus by the end of the first semester we had accomplished a reasonable mastery of the fundamental processes of arithmetic; had acquired about half the subject-matter of algebra outlined for a normal section; and had gained the foundation of a mathematics vocabulary.

Methods of instruction, as well as subject-matter had to be adapted to the needs of the slow groups. The class hour was not so much a period to recite as a time for teacher and pupils to work together. Much oral work was used for it made possible the correction of misconceptions and mistakes before they became fixed by practice and it economized time in drill.

Drill for a particular type of problem was not only intense for the given time period, but it was repeated at short intervals throughout the semester. One of the teaching problems was to give these "slow-movers" as much drill as they needed and yet make it enjoyable to them. Again, competition and praise were the incentives which were most effective. A score sheet on the bulletin board to show which tests in the exercise book had been completed without mistakes, was the means of sending the pupils to their seats before the class hour began, to see which problems they had missed and to correct the same.

The approach to problems was made through a review of the representation of ideas by algebraic symbols. The attack upon a new type of problem was begun by the silent reading of a problem, followed by class discussion of it to reveal faults of incorrect reading and misinterpretation. After the pupils had analyzed the situation and had seen the problem in it, they dictated the representation of the quantities involved while the teacher expressed them on the board. Then, either a pupil or the teacher at the dictation of a pupil, showed the relationship among the quantities by writing the equation, which all then solved.

Progress tests consisting of a problem or two and containing as few processes as possible were given almost daily. These served either to show that the pupils were comprehending each new step or to reveal specific weaknesses so that the teacher might decide intelligently upon the type of remedial instruction.

The homework for slow sections was, of course, less extensive than in normal ones. Problems were not assigned until the teacher had objective evidence of the pupils' ability to solve them. Sometimes this was accomplished by the pupils working at the board and at other times by seat work. The members of the class preferred the latter plan, especially if they discovered that they had worked a few of the problems to be assigned, for they felt they were started on the next day's preparation.

I have explained the working out of this plan with reference to the 9B group in detail because it is there that we feel we have the greatest problems. Pupils who return to high school the second year are a more select group; consequently fewer of them are assigned to slow sections. Only the most persistent return for a third year of algebra. Almost without exception, they do quite satisfactory work for they must have a reasonable amount of ability to have reached this stage, and conscientiousness does the rest. Previous work in algebra is used as the basis of grouping after the first semester.

I shall now try to anticipate some of the questions which may be raised regarding the plan.

1. "Is it worthwhile to spend so much time on algebra?" is the first question I should expect anyone to ask. Frankly, "I do not know." As I look at the report cards of the pupils who are enrolled in our slow sections in algebra, I wonder what we could attempt to teach them in order to have the results justify the time spent, for their slowness is not confined to algebra. With very few exceptions, they are failures in other subjects. Our curriculum has little to offer that is within their ability, and to introduce new courses to fit their needs will take time. In the interim, the best we know to do is to segregate them, adapt our present courses to their mental levels, and give them time to realize their possibilities.

2. "Will not this plan encourage loafing since if a pupil of normal ability finds his way into a slow section, he can make a credit without effort?" Doubtless it would under such conditions. However, it is the teacher's business to see that no one of normal ability is assigned to a slow section and our teachers have attended to it effectively. I have found no pupil in a slow section who could have done the work in a normal one.

3. "Does the plan necessitate more teachers?" It has not with us. While slow sections should be small, normal sections can be enough larger to balance the situation. Where we had sections of twenty-five before, we now have about twenty in a slow section and thirty in a normal one.

4. "How do teachers feel about being assigned slow groups?" It goes without saying, that no teacher prefers a slow group to a normal one. However, slow pupils are ever with us and have to be taught. It is only a question of whether they are to be

taught in mixed groups or segregated sections. Since no teacher is given more than one slow section, whatever of extra work and patience it may exact is compensated for in the satisfaction of teaching other sections from which the dull pupils have been removed.

5. "Are pupils humiliated by being assigned to slow sections?" When they are first segregated they do feel somewhat abased, but after they have had time to appreciate the advantages to themselves, most of them prefer such an arrangement.

6. "What is the status of the pupil who transfers from us to another school while his unit of algebra is incomplete?" If he enters a school which does not have a classified system, he will no doubt have to start at the beginning of the semester's work upon which he was engaged. However, he will be no worse off than if he had taken a record of failure from us, as he must otherwise have done.

I should prefer not to call attention to the defects of this plan, but I feel constrained to do so. As I have already said our homogeneous groups in 9B have not been organized before the end of the first month. The purpose in this delay has been to permit pupils to be oriented in high school surroundings before being subjected to the placement test and to give the teacher time to make the acquaintance of pupils in order to pass judgment upon them. This time for segregation has the following undesirable results:

1. When the shifting takes place, about fifty per cent of the pupils have to change teachers. This is objectionable both from the standpoint of the pupil and the teacher for, other things being equal, the better they know each other the greater the chance for the pupil's progress.

2. It is sometimes necessary to make changes in pupils' programs in other subjects than algebra in order to keep sections of workable size.

3. It makes additional work for the principal since all of the transferring is done by him.

4. It entails extra work for the teachers in making new class records and seating charts.

5. It adds to the duties of the clerks in the office for they have to change the program cards of the pupils concerned.

Thus, though this feature may be desirable in some respects, it is nevertheless a labor-increasing factor.

The fact that "teacher's judgment" is a factor in determining placement has caused some difficulty. It is a tremendous responsibility to say after four weeks study of him, that a pupil must double the time spent on a course, since a mistake can not be remedied for a year, and some of our teachers are loathe to pass such judgment.

As to the desirable results accomplished by this plan:

1. It has enabled the bright pupils to forge ahead so that they have not only completed the subject-matter outlined in the course, but they have taken much supplementary material.

2. The possibility of being dropped back to slow sections has served as a spur to lazy pupils in normal classes.

3. It has reduced the number of failures. A smaller percentage of our freshmen failed in 9B algebra than in another required subject where they were not so classified. Usually the percentage of failures in 9B algebra is much higher than in 9B English, but under this plan there is little difference between the two. Of course, it does not eliminate failures for there are non-intellectual factors, such as attitude and effort, necessary to success. Laziness, indifference, and irregular attendance spell failure under any plan.

4. It has made possible the giving of slow pupils as much explanation as they needed for real understanding, without boring bright pupils with explanations of what they already understood.

5. It has lessened the fear of slow pupils to ask questions and volunteer information.

6. It has helped in developing habits of industry in slow pupils. Under the unclassified system, there was little incentive to work since their supreme efforts could not earn passing grades, but now they feel repaid for application.

7. It has made for greater honesty. Inability to compete with superior pupils under the plan of mixed grouping, increased the temptation of slow pupils to copy work in order to appear the equals of their classmates.

8. It has developed a happier frame of mind in slow pupils since they have found work which they are capable of doing.

To summarize, our experiment has consisted of separating the pupils who enter high school into ability groups at the end of the first month, on the basis of an arithmetic test and teachers' judgments. In 9A and 10B, the segregation has been determined by

previous work in algebra. The normal groups have been expected to accomplish the work outlined in the course of study plus as much supplementary material as possible. The subject-matter for a slow group has been selected by its particular teacher and methods of instruction have been modified by her to meet its needs. Courses for slow groups have extended over double, or nearly double, the time period of a normal section. The plan has caused extra work at the time of segregation, but this fault is negligible as compared with the benefits which have been derived. Bright pupils have been stimulated to the realization of their possibilities, and those who are slow but conscientious in their plodding, have achieved better results and have enjoyed the fruits of their labors.

NOTICE TO MEMBERS

Members of the Council whose subscriptions to THE MATHEMATICS TEACHER expired in May, 1929, have been sent two renewal notices to which some have not responded. In order not to cause a break in their files these members are being sent the October number, but their names will be dropped from the mailing list and from the register of members, which is about to be issued, unless their dues of $2 are sent in at once.

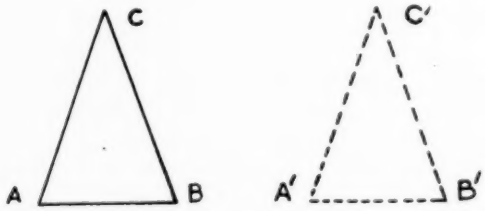
PROVING THE EQUALITY OF THE BASE ANGLES OF AN ISOSCELES TRIANGLE

By JOSEPH A. NYBERG

Hyde Park High School, Chicago

In a recent paper (A Different Beginning for Plane Geometry, by H. C. Christofferson, MATHEMATICS TEACHER, Dec., 1928) much emphasis is placed on the fact that the existence of an angle bisector must be assumed in proving that the base angles of an isosceles triangle are equal. Because of this assumption it is argued that we could better begin by assuming triangles congruent if the corresponding sides are equal. The following is a proof of the equality of the base angles of an isosceles triangle without the use of the angle bisector. Since the quqestion seems important I present the proof with some detail. Step 4 is the significant step.

Given: $\triangle ABC$ with $AC = BC$. Prove: $\angle A = \angle B$.



1. Construct $\triangle A'B'C'$ making $\angle C' = \angle C$, $A'C' = AC$ and $B'C' = BC$.

2. $\triangle ABC$ is congruent to $\triangle A'B'C'$. (Two sides and the included angle, etc.)

3. Then $\angle A = \angle A'$ and $\angle B = \angle B'$. (Corresponding parts on congruent triangles, etc. $\angle A$ and $\angle B'$ are really *corresponding* angles because they are respectively opposite the equal sides BC and $B'C'$.)

4. Place $\triangle A'B'C'$ so that C' is on C and $B'C'$ on AC, *not* on BC. (This is possible because $B'C' = BC = AC$. See step 1 and the hypothesis.)

5. $A'C'$ will fall along BC because $\angle C = \angle C'$.

6. A' will fall on B because $A'C' = AC = BC$. (See step 1 and the hypothesis.)

7. Then A' is on B, and B' is on A. Hence $A'B'$ coincides with AB.

8. Then $\angle C'A'B' = \angle CBA$. (The definition of equal angles: their sides coincide.)

9. But $\angle C'A'B' = \angle A$ in step 3.

10. Hence $\angle A = \angle CBA$. (See steps 8 and 9.)

I make no claims to originality in this proof. In fact in my classes in solid geometry when the usefulness of auxiliary triangles is seen in connection with symmetric spherical triangles I mention that an auxiliary triangle can be used to prove the above theorem. Invariably some pupil finds the above proof and is ready to present it the next day.

The question then arises: Since this proof will give a more logical structure should we use it in place of the proof involving the angle bisector. The logician may say "No" on the ground that the above proof necessitates turning a triangle over through three-space. Teachers with experience in the classroom (not bothered by consideration of three-space) will say "No" for much better reasons.

One objection to the above proof is its cleverness. The pupil could not have found it by himself using what he has been taught previously about congruent triangles. Further, the traditional proof is a splendid example of the usefulness of congruent triangles. If CD is the angle bisector, the three equations used in the traditional proof are

$$AC = BC, \qquad \angle ACD = BCD, \qquad CD = CD.$$

The three equations discovered by using the equality of the corresponding parts are:

$$\angle A = \angle B, \qquad AD = BD, \qquad \angle ADC = \angle BDC.$$

The second and third equations also lead to important discoveries (not found in the proof in this article). The pupil feels that with a very slight amount of work (proving two triangles congruent) he has made *three* discoveries. Usually when two triangles are proved congruent we are interested only in one pair of corresponding parts, but here all three are valuable. In many classes the thrill to be derived from this one simple piece of work justifies the entire first month's work and arouses an enthusiasm that can never be obtained by increasing the number of assumptions. This argument will not appeal to a logician, but the classroom teacher is interested chiefly in the pupil.

THE NEED FOR TESTING PRACTICES IN CURRICULUM REVISION

By WM. HERBERT EDWARDS

Northeastern High School, Detroit, Michigan

Anyone engaged with the problem of curriculum revision is likely to be very distinctly impressed by the literature on the subject where three dimensions are essential to redesigning a curriculum. The three predominately mentioned are first, aims, second, materials and third, methods of teaching. The purpose of this paper is to call attention to the significance in mathematics curriculum revision of a fourth dimension, namely the testing practice at present employed in the mathematics classroom.

1. *Testing for Mastery.*—There are at least two testing practices at present employed in mathematics classrooms that may affect the success or failure of a revised curriculum. One of these practices is that of testing for mastery of skills regardless of any other possible aim. For example, one of the outstanding aims, if not the major objective of ninth grade mathematics in Detroit for the majority of pupils, the non-college preparatory group, is that this year's work shall provide an exploratory overview of some of the general fields of mathematics. Yet committees of teachers representing the various schools and a committee in one of the high schools never mentioned the major objective but considered only ways and means of testing mathematical skills.

The significance in curriculum revision of the fact that mastery of skills is being tested for is that this testing habit indicates that the major objective operating in the classroom is mastery of skills. It has been pointed out that the major objective for the majority of ninth grade mathematics pupils in Detroit is not mastery of skills but exploration. Yet, an analysis of lesson plans of teachers of ninth grade mathematics in one of the schools revealed that in every case either in a lesson where a new concept was to be treated or where a review was planned for, the only goal of the teacher making the plan was mastery of skills. Hence, anyone revising a mathematics curriculum is faced with the responsibility of guarding against the probability at present that the subject matter will be misused in the classroom.

There is the possibility that the content of a unit of instruction might be more intelligently used if it were known that pupils were to be tested by an instrument that measured or indexed the aims for the unit. If the aims were then stated, instruction could be guided thereby. The fact that the aims were to be tested would further increase the probability that the intended use of the subject matter would be attempted in the classroom. It is up to the curriculum builder to supply these aims and these tests for each unit of every course.

TABLE I

FREQUENCY OF SCORES OF TEST ON ARITHMETIC FRACTIONS BEFORE AND AFTER STUDYING ALGEBRAIC FRACTIONS

Before		After	
Score	Frequency	Score	Frequency
30	7	30	6
29	3	29	9
28	6	28	6
27	1	27	2
25	2	26	1
24	1	24	1
22	1	17	2
19	1	16	1
15	1		
13	2		
12	1		
11	1		
5	1		
	28		28

2. *Testing After Study.*—The practice of testing at the completion of a unit of study instead of before and after the unit is the second present day testing habit employed in mathematics classrooms that the curriculum builder must recognize. Need evidence be cited in support of this characteristic of present day testing?

The significance in curriculum revision of this practice of testing only after completion of a unit of study may best be illustrated by an interpretation of the results of a test given before and after the unit on algebraic fractions. One of the aims for the unit was to develop facility with arithmetic fractions of denominators no higher than 12. Hence, the test com-

prised five examples each in adding, subtracting, multiplying and dividing two common fractions and in adding and multiplying mixed numbers, no denominator being larger than 12. The test results are summarized in Table I.

An analysis of the score frequencies made after the learning unit reveals that 82 percent of the pupils made a score of 27 or better. Such a score represents 90 percent of the highest possible score on the test. That this is no index of the influence of the unit on the achievement of the tested aim for the unit is evident from an inspection of the data from the test given before the unit was studied. These data show that at that time 60 percent of the pupils could already make a score of 90 percent or better on this test. Hence, it should be evident that in order to measure the extent to which any aim for a unit is being realized, tests must be given both before and after the pupils attempt the unit.

The significance for the curriculum builder of the present practice of testing only at the end of a learning period is two-fold. In the first place unless tests are provided at the beginning and end of each unit of instruction there is no control over individual differences in the extent to which unit aims are already realized in the class. In the second place, the curriculum builder has no basis for comparing the relative efficiency of different subject matter in achieving given aims unless statistical determinations of gains made between tests are possible.

Conclusion

1. Two testing habits characteristic in the mathematics classroom were pointed out. These were first, testing on mastery of skills and second, testing at the end of a learning period rather than both at the beginning and the end.

2. Evidence was introduced to support the suggestion that unless tests are provided before and after each unit of instruction by the curriculum builder three serious dangers to the success of the curriculum are almost certain to arise. These dangers are first, the curriculum will be misused, second, there will be no provision for discovering those pupils who have already achieved one or more aims for the unit, and third, no provision will exist for measuring the extent to which the material in the unit contributes to the realization of the aims of the unit.

THE PLACE OF MATHEMATICS IN JUNIOR HIGH SCHOOL EDUCATION

By AGNES ROWLANDS

Jamaica Training School, New York

Before discussing the place of junior high school mathematics in education, it will be well to decide what criteria to use in determining whether or not a subject or a method has educational value. According to the criteria set up in the second year book of the National Council of Teachers of Mathematics, a subject or a method may be said to have educational value:

1. If it helps the student in the exploration of his own mind; the more important the guidance afforded, the greater the educational value.

2. If it helps the student to understand the world; the more important the segment of human knowledge considered, the more directly the new understanding functions, the greater the educational value.

3. If it establishes correct habits and attitudes; the more important the habits and attitudes developed, the greater the educational value.

The teaching of mathematics by good teachers is to-day characterized by common sense. The pupil and his interests, and his assets, intellectual and social, which he brings to school, are the center. The value in teaching algebra lies not simply in knowing algebraic processes, but in what has been called an "attitude established, and in the intellectual experience gained." Algebra has very little value for most pupils unless it develops "the understanding attitude and habit of mind."

In life an understanding habit of mind means that when one is confronted with a new situation, he is able to reason correctly about it. He collects his facts, tabulates them, mentally at least, so he may better analyze them. He constructs an hypothesis, tests it, then perhaps rejects it, and constructs another hypothesis. Then if he keeps this hypothesis he seeks methods for action in accordance with it. When confronted in life by a new situation, one does not ask for the rule or procedure. There

could be no better means for developing this understanding attitude of mind so valuable in meeting new situations in life than is afforded by a proper study of junior high school mathematics.

When I was teaching algebra in the junior high school in the days before the adoption of the modern junior high school course with its work in intuitive geometry, I prefaced the work in algebra with some work in intuitive geometry. Through this work in intuitive geometry I was able to lay a foundation for the understanding attitude in algebra. The pupils in experiments performed by each one, discovered a number of the simpler geometric theorems. For example, they tried to find out the relation of the hypotenuse to the shortest side in a right triangle. The pupils did this by drawing three right triangles of different sizes all having the other two angles, 60° and 30°. They next measured the length of the hypotenuse, and shortest side in each triangle, using metric rulers. They then formed a tentative rule, that "in a right triangle having one angle equal to 60° the hypotenuse is twice as long as the side opposite the 30° angle." Then the pupils tested this tentative rule by drawing a fourth right triangle, having one acute angle of 60° They measured the hypotenuse and shortest side, to see if the hypotenuse was twice as long as the side opposite the 30° angle. Finding that their tentative conclusion checked, they now called their tentative conclusion correct. After the pupils had performed a number of such experiments in intuitive geometry, they were asked to write an essay on the procedure to be used, in trying to solve a problem, where the solution of the problem involved a discovery of the relation between two quantities. These essays were sometimes crude because of lack of knowledge of technical terms the use of which would have simplified their English. I then supplied these technical terms, and the pupils rewrote their essays. The procedure they formulated was of course the one with which you are all familiar; known as the laws of formulation, namely,

1. The problem, calling for the discovery of a relation between two quantities.
2. Simple experiments by each pupil, giving him at least three cases, to use in discovering the relation.
3. Tabulation of the cases.

4. Studying tabulated data to discover likenesses or differences that hold in each case, so as to formulate the tentative rule.
5. Formulation of the tentative rule, expressing the relation between the quantities.
6. Checking the tentative rule.

We then discussed in class the possibility of using this procedure when confronted in life with a new situation. Pupils gave situations in life where this procedure could be used, we discussed the use of this procedure by the scientist. We saw that it is because the scientist uses it in his work it is called "the scientific method." We discussed the social value of such a procedure, especially the habit of not jumping to a conclusion from a single case, and of refusing to accept a conclusion that appeared to be correct until its correctness had been shown. We discussed concrete cases where people with the understanding habit of mind, and understanding of the scientific method with its checking, habit, had been able to prevent serious social wrong, that would have resulted if permitted to stand unchallenged, a conclusion reached from one instance, or a conclusion not carefully checked to determine whether or not it was a correct conclusion

Then we took up the work in algebra. *Each pupil by himself* was expected to discover the rule for himself, by using the scientific method. For example, suppose in trying to use a formula we had met a need for knowing how to multiply with exponents, the definition of an exponent being understood. No work was done at the blackboard. *Each pupil* at his seat sought to discover the rule, by using his knowledge of the scientific method.

Problem.—To discover how to find the product of like letters, with exponents.

Collecting cases.—Each pupil worked several examples such as the following:

(1) $a^3 \times a^2 = ?,$
$a^3 = a \cdot a \cdot a,$
$a^2 = a \cdot a;$
$\therefore \quad a^3 \cdot a^2 = a \cdot a \cdot a \cdot a \cdot a = a^5$
or $a^3 \cdot a^2 = a^5.$

(2) $c^4 \cdot c^3 = ?,$
$c^4 = c \cdot c \cdot c \cdot c,$
$c^3 = c \cdot c \cdot c,$

$c^4 \cdot c^3 = c \cdot c \cdot c \cdot c \cdot c \cdot c \cdot c = c^7,$
$c^4 \cdot c^3 = c^7.$

(3)
$$x^5 \cdot x^2 = ?/,$$
$$x^5 = x \cdot x \cdot x \cdot x \cdot x,$$
$$x^2 = x \cdot x,$$

$$x^5 \cdot x^2 = x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x = x^7,$$
$$x^5 \cdot x^2 = x^7.$$

Tabulating Cases:

$$a^3 \cdot a^2 = a^5,$$
$$c^4 \cdot c^3 = c^7,$$
$$x^5 \cdot x^2 = x^7.$$

Tentative Conclusion.—Add the exponents of like letters in each expression to obtain the exponent of those letters in the product.

Checking the Tentative Conclusion:

$a^4 \cdot a^2 = a^6$ by tentative rule,
$$a^4 = a \cdot a \cdot a \cdot a,$$
$$a^2 = a \cdot a,$$

$$a^4 \cdot a^2 = a \cdot a \cdot a \cdot a \cdot a \cdot a,$$
$$a^4 \cdot a^2 = a^6.$$

The tentative conclusion checks. We therefore have

The Rule.—Add the exponents of like letters in each expression to obtain the exponent of those letters in the product.

The pupils carried out this work because of their knowledge of the scientific method. Once the purpose was formed, the planning, executing and judging, were the work of the pupils themselves.

The bright children of course always discovered the rule, and checked the correctness of their conclusion, some time before the others. As soon as a pupil had brought me his conclusion, and had shown me his work of checking this conclusion and I saw his work was correct, I either gave him drill in using the newly discovered rule, or permitted slower pupils having difficulty in discovering the rule, to go to him for help, instead of coming to me. There were several results of this kind of work. One result was that the pupils developed an understanding habit of mind in their work in intuitive geometry and algebra, and carried over this understanding habit of mind to new situations that confronted them.

Psychologists tell us there is transfer of training where there

are common aims, common elements, and common methods. And there are such common aims, common elements, and common methods, in thus solving a problem in intuitive geometry or algebra, and in solving a problem in life, when one is confronted by a new situation. Algebra, when taught in this way, does not simply give a pupil skill in using a tool, which the pupil takes up at will, and lays down at will. When algebra is taught in this manner it becomes a means of education. The pupil thus trained in his algebra is acquiring a *mode* of *thought* which will change all his later mental operations. The pupil is no longer capable of returning to the level of loose, inexact thinking that characterized his earlier methods of viewing the world. Says Professor Judd, "Pervading all one's thinking about the world should be the general attitude that the objects that come into experience can be arranged and classified. Order, arrangement, classification, are products of the mind's reaction on the world. It is the business of the school to transmit to the pupils the intellectual methods of arrangement by which the complexities of the world may be unraveled, and a new pattern made of experience. The most comprehensive and flexible patterns for the rearrangement of experiences are those supplied by the mathematical sciences." Just such work in intuitive geometry and algebra as I have been describing, transmits to the pupils the intellectual methods of arrangement by which the complexities of the world may be unraveled, and a new pattern made of experience. This method of teaching algebra also helps the pupil in the exploration of his own mind. It establishes the understanding attitude and habit of mind. It therefore has educational value. Algebra is not something the pupil simply knows how to do. The work in algebra has resulted in an "attitude established, and in intellectual experience gained." The pupil can not only repeat the rule, and use it. The pupil can repeat the explanation, the derivation of the rule. In developing power to do this there has been real education, for the pupil has been developing habits of exact thinking, which will make it impossible to return to the level of loose, inexact thinking.

There is also another great value of algebra, when studied in this way. This value is, the attitude toward algebra itself, that the pupil obtains. In the past how often pupils have disliked algebra. What a far cry from this is the attitude toward algebra

illustrated by an experience of a teacher who told me that she changed from a "do as I tell you" type of algebra teacher to the type of teacher who guides her pupils in their discovery of rules for themselves. A certain boy was a real discipline problem, under the "do as I tell you" regime. When this same boy learned how to discover rules in algebra for himself, he used to come to school early in the morning, and ask his teacher to give him something more to discover in algebra.

I have said that a subject or method has educational value, if

1. It helps the pupil in the exploration of his own mind.
2. If it establishes correct habits and attitudes.

We have seen thus far that the study of intuitive geometry and algebra in the junior high school does have educational value because it helps the pupil in the exploration of his own mind, and because it establishes correct habits and attitudes.

One of the most thrilling illustrations of the way a proper study of junior high school mathematics helps the pupil understand the world, is the fact that a study of intuitive geometry and of formulas in algebra will make it possible for a pupil to understand the story of how a mathematician discovered the existence of a planet 2,800,000,000 miles from the earth. It is of course a familiar story to many of you, of how Sir William Herschel first established the fact that a heavenly body hitherto regarded as a star was really a planet, which was now given the name Uranus. Astronomers at once set to work to compute the orbit of Uranus. But the planet deviated from the orbit computed. The difference between the actual and computed places of the planet amounted in 1845, 56 years after the first computations, to the intolerable quantity of nearly two minutes of arc. It became certain to astronomers that some unknown force must be acting on the planet. A young English mathematician, a graduate of Cambridge University, solved the problem, and deduced the hitherto unknown planet's approximate place. Adams communicated his discovery to Professor Challis of Cambridge University. Challis used the laborious method of observing the positions of all the stars in the suspected region, and did not map his observations as fast as they were made. In the meantime, Leverrier, a young French astronomer, solved the problem. Galle, an assistant in the observatory at Berlin, had just completed a map of the region of the heavens in which

Leverrier's calculations indicated that the planet was to be found. To Galle, Leverrier wrote a letter which read in substance "Direct your telescope to a point on the Ecliptic in the constellation of Aquarius, in longitude 326°, and you will find within a degree of that place a new planet, looking like a star of about the 9th magnitude, and having a perceptible disk."

Leverrier, like Adams, solved the problem by using Newton's famous formula for the law of gravitation,

$$G = \frac{MM'}{D^2},$$

where G represents the amount of attraction between two bodies whose masses are represented by M and M^1, and whose distance from each other is represented by D.

The planet was found at Berlin on the night of February 23, 1846, in accordance with this prediction of Leverrier, within half an hour after the astronomers began looking for it, and within 52′ of the precise point that Leverrier had indicated.

Surely there could be no more wonderful story to illustrate the way a study of junior high school mathematics helps the pupil to understand the world. His study of angles and arcs and ellipses in intuitive geometry enables the pupil to understand what is meant by the statement, "the difference between the actual and computed places of the planet amounted in 1845 to the intolerable quantity of nearly 2 minutes of arc." A study of angles enables the pupils to understand the fact a large angle represents the difference in direction between two nearby bodies very close together, while a very small angle represents the difference in direction between two distant bodies that may be very far apart. So the pupil from his work in intuitive geometry realizes why nearly 2 minutes of arc was an "intolerable quantity" to the astronomers. For at the distance of the planet Uranus, a departure of nearly two minutes of arc from the planet's computed orbit, would mean a deviation of many, many miles. The pupil, as a result of his study of intuitive geometry, appreciates the wonderfulness of the statement, "that the planet was discovered within half an hour after the astronomers began looking, for it, within 52′ of the precise point that Leverrier had indicated."

Finally, by his study of formulas in algebra, the pupil can

interpret the formula used by both Adams and Leverrier in solving the problem, namely,

$$G = \frac{MM'}{D^2}.$$

The pupil realizes this means "the amount of attraction between two bodies is directly proportional to the product of their masses, and inversely proportional to the square of the distance between them."

Thus does a proper study of mathematics in the junior high school help the pupil to understand the world. And a subject or method which helps a student to understand the world has educational value.

But such a story as this one of the mathematical discovery of Neptune does more to help the pupil understand the world through his knowledge of geometry and of algebra. It affects the pupil's attitude toward his further study of mathematics. A formula becomes what Maria Mitchell, the great Vassar astronomer, called it, "a hymn of the universe." The pupil realizes that when Newton discovered this great formula

$$G = \frac{MM'}{D^2},$$

Newton was really thinking the Creator's thoughts after Him, for this formula gives one of the secrets in accordance with which the Creator chose to build His universe. "A thorough advocate in a just cause, a penetrating mathematician facing the starry heavens, both alike bear the semblance of divinity."

Indeed a proper study of mathematics should not only help the student in the exploration of his own mind; should not only establish correct habits and attitudes, but should give the pupil an understanding of the world, that will awaken in him reverence, and a greater appreciation of some of the essentially religious values of life.

One of these essentially religious values that may be better appreciated through a study of mathematics is our infinitesimal nature. We have been talking about the mathematical discovery of a planet 2,800,000 miles from the earth. But the pupil's study of mathematics enables him to appreciate something of the tremendous significance of the fact that while light travels 11,000,000 miles a minute, and can come from the sun in 8½ min-

utes, it takes light over 1,000,000 years to come from some of the spiral nebulae. "Our hopes, our fears, are," to quote David Eugene Smith, "all so trivial, so infinitesimal, so like a lost electron in our solar system, as compared with our relative importance in the universe as revealed to us by the calculations which mathematics brings to bear upon the great problem."

Again a study of mathematics brings its devotees into continual contact with the infinite. "To the mathematician, the laws of the finite give way without question, when he enters the domain of the infinite." Parallel lives may not meet in finite space; but they do meet at infinity. "In the domain of the infinite," as Dr. Smith says, "the part may indeed be equal to the whole, in spite of the childish beliefs of our finite minds."

Again in mathematics we deal with eternal laws. $(a+b)^2 = a^2 + 2ab + b^2$, that law always was true and always will be true. So a proper study of mathematics may be of great educational value because it increases our understanding of and appreciation of certain essentially religious values. Dr. Smith makes clear this great educational value of a study of mathematics in the following way:

1. The infinite exists.
2. Immortal laws exist.
3. The laws relating to finite magnitudes do not hold respecting the infinitely large or infinitely small.
4. No factor is ever lost.
5. Time may be a closed curve.

Put into theological language, these mathematical truths become

1. God exists.
2. God's laws exist.
3. God's laws are entirely different from ours.
4. The soul exists and is eternal.
5. God looks at time as a whole.

Finally, the real mathematician, as Novalis said, is an enthusiast, per se. A study that makes a person an enthusiast, surely has educational value.

May I conclude with a quotation from Weierstrass, who has said, "It is true that a mathematician who is not somewhat of a poet will never be a perfect mathematician." Mathematics is the foundation of great music and great art. The true mathe-

matician always loves beauty and is at heart something of a poet. Only the poet at heart sees life steadily and sees it whole. So a proper study of mathematics is of educational value because it helps the pupil to understand the world in such a way as to develop in him an enthusiasm for life, and an appreciation of essentially religious values. It gives a greater appreciation of beauty, and makes of those who study it properly, poets at heart, with power to see life steadily, and to see it whole.

ERRATA

In Professor Judd's article in the April 1929 number of THE MATHEMATICS TEACHER, certain important words were omitted in printing. If after the word "comes" at the end of line 5 on page 193 the following words are inserted the passage will read as it should:

"into the possession of one of the most universal of ideas. The mind can now, being the source of the identical element, put this element into whatever it will. Transfer there is. It is that"

TEACHING GEOMETRY INTO ITS RIGHTFUL PLACE*

BY J. O. HASSLER

Norman, Oklahoma

During the last ten years there has been much concern over geometry's losing its place in the high school curriculum and a consequent loss by high school pupils of a vital element in their educational training. Attacks by educationists on the hitherto impregnable position of geometry in the curriculum has drawn the fire of its ardent supporters. Organizations of teachers have sent out questionnaires to citizens in all walks of life to get evidence on the value of mathematics in their professions and business. Much has been written about the educational value of mathematical training. We have "boosted" mathematics to such a degree by saying nice things about it that a "high-powered" chamber of commerce might well take lessons of us.

These efforts to hold geometry in the curriculum remind me of a typical method of treating headaches; namely, that of taking bromides for temporary relief. A saner method of treating habitual headaches is to get at the source of the trouble. So it seems to me that instead of our deadening the pain of a decreasing demand for geometry by bromidic statements about the value of geometry, we should investigate our methods of teaching.

We are living in an age of strenuous competition; an age of chain stores and high pressure salesmanship. The business man stimulates interest in his goods in order to create a greater demand for them. The teacher assumes that the market for his products is established and his only care is to supply the demand created by an enforced curriculum. It reminds me of an incident of history wherein a powerful nation across the sea once sold tea to its colonists and then laid a tax on it. Then came the Boston Tea Party. We should not be surprised to have a whole shipload of mathematics dumped into the harbor of discarded subjects as this or that state takes geometry off the list of required studies in the curriculum.

* Read before the Meeting of the National Council of Teachers of Mathematics at Cleveland, Feb. 23, 1929.

My opinion is that dull and mechanical teaching is solely to blame for the efforts on the part of some to remove geometry from the curriculum. In their ignorance of the true cause (poor teaching) they lay the blame on the subject itself. It is imperative that the teacher acknowledge a responsibility to produce goods that will create a demand for themselves. High powered salesmanship and exaggerated advertising may stimulate business temporarily, but in the end, unless the product comes up to specifications, they leave a merchant in worse shape than if he ignored the methods of creating a demand. Let us take warning and give more attention to methods and less to propaganda.

I think we are giving methods more attention in the National Council, which exists primarily for the exchange of ideas. I may not tell you anything you have not thought of yourself, certainly not if you are a good teacher. My way of saying it, however, may challenge your attention and make an impression.

I want to discuss briefly a few points about teaching geometry as it deserves to be taught, so that it may function best in the lives of the pupils and thereby win the place of honor it should have in the minds of all educators, as it has found and filled its place in the development of civilization. I shall recommend no "pet" methods of my own. I do not pretend to discuss all phases of good teaching. The things I say have been said before in a multitude of ways; but it seems to me they should be said again and again. On the basis of the assumptions I have made, I shall develop my theme in traditional geometric style with a proposition, three corollaries, and a scholium.

The proposition. *The best efforts of the teacher throughout the course should be spent to secure active, self-motivated, pupil-participation, so as to develop habits of independent thinking.*

Let us consider first *the unconscious attitude of the teacher* toward the presentation of the subject matter. Is it an attitude of *giving out* or *drawing out?* The etymology of the word *educate* suggests literally a "leading out" or "drawing out." How far some of us have departed from the original purpose of education as indicated by the word! Too many of us act at times as if we were dispensary clerks, giving out knowledge without asking for any purposeful thinking effort on the part of the students. We give a "free handout" instead of demanding that the recipients "saw wood" for it. Let me illustrate.

Suppose that the theorem is "The angles opposite the equal sides of an isosceles triangle are equal." A bisector of the vertex angle is needed. We should not tell the pupil—the text should not tell the pupil—at the outset to "draw a bisector." He should be led to discover for himself that a bisector is needed in order to prove the congruence of the two triangles into which the isosceles triangle is divided so that he may compare the corresponding angles and thus prove the base angles are equal.

Again, a pupil asked concerning an original exercise just proved in class, "How do you know those two triangles are congruent?" The teacher promptly replied, "Because the three sides of one are equal to the three sides of the other," thinking he was doing the pupil a service. He should have led the pupil to discover the reason for himself. It might have been done with two questions; for example, "What are the necessary conditions for congruent triangles?" and "What corresponding parts of these two triangles do we have equal?"

The unconscious reaction of a teacher when a pupil asks for information is a vital element in successful teaching. Unless our sole purpose in teaching geometry is to get into the minds of the pupils certain facts, then a prompt answering of the pupil's question is not the best thing to do. No teacher will admit that as his sole purpose in teaching geometry. We have as one of the principal objectives in teaching geometry the development of habits of independent thinking rather than the memorization of certain connected facts and logical methods of proof. This aim should be uppermost in the mind of every teacher. It should influence his teaching to such an extent that his inevitable reaction to any question is to ask another which leads the pupil to discover what is wanted. Perhaps this can not always be done. In such cases the attitude should be that of helpful suggestion. When geometry is taught in such manner the pupil is given the best possible training in developing habits of independent thought; for habits are strengthened by repetition of the act, and nowhere in the whole range of high school subjects are there more abundant opportunities for the act of independent thinking than in a course in geometry well taught.

There are few qualities of mind more seriously needed in our national life to-day than that of independent thinking. We need it in our politics, we need it when we open a lurid pros-

pectus describing oil royalties or mining stock for sale. We need it in our business life and we need it in our religious life. We are too prone to rely upon the truth of anything we see in print without checking up on the conditions governing the situation as presented, and, with the use of some independent thought, making some conclusions of our own. If through the medium of geometry we can encourage a student to develop the habit of relying upon his own initiative, of forming his own conclusions on the basis of conditions set forth, which he must inspect carefully, we have taken a long step toward making the subject function in the life of the pupil. We are on the way toward teaching geometry into its rightful place.

Last week I heard a repetition of an old, old story. A thoughtful university student who intends to study law came to me with the statement that he wanted to major in mathematics for his arts degree. I said, "Why not in government?" His reply was that he was getting no training in systematic and logical reasoning in his courses in government. A successful lawyer had told him that in working up a brief there was so much of similarity to the type of work done in geometry that he considered that as one of the most useful courses he had taken in school. That lawyer's testimony is a tribute to his mathematics teacher. He might have been taught by any one of thousands of teachers and never have recognized the similarity. He would have been kept so busy memorizing proofs he could not have had time to think. His plight would have been like that of the countryman who "couldn't see the town for the houses."

Some will take issue with my next statement. I never could see the value of having students copy proofs of original exercises in a notebook. When the exercise is proved the value is received by the pupil. I do not want him to try to remember that proof. If I should ever ask him again to prove it I would expect him to go through the steps by independent thinking. If we quit making pupils *learn proofs* and teach them how to *think,* giving them inspiration all the while by our own attitude, we shall help to win for geometry the place it deserves.

Corollary 1: *It follows from our proposition that we must make extensive use of the practical applications of geometry.*

So much has been written on this topic that I shall make but

a few statements. Many a student has been able to recite the proof of a theorem perfectly with a complete understanding of the connection between statements and reasons and yet not be able to realize any connection between those abstract geometrical relations and some real problem involving them. Suppose we have proved that if the two diagonals of a parallelogram are equal the figure is a rectangle. Does the pupil after learning to prove that know enough to measure the diagonals of a rectangular tennis court he has just laid out? There has been no text published yet with enough exercises of the practical type. Applications found by pupils in the locality are valuable. Making the application to some concrete situation may call forth in the pupil the same independent thinking that is required in proving theorems—depending on the way he is taught. For this reason we may have no scruples against omitting scores of the usual exercise-theorems of pure geometry, which have no value except that they furnish the pupil opportunity to indulge in various methods of proof, and replace them by exercises in the applications of geometry. But, *let the pupil find the application* whenever possible instead of pointing it out to him.

Let us not forget that the use of practical applications in teaching produces several valuable by-products aside from the aid in reaching the main objective as given in the main proposition.

Corollary 2: *We can do much to vitalize the subject of geometry and encourage independent thinking by emphasizing the functional relation.*

A great deal has been published on this topic in recent years. In Chapter VII of the *Report of the National Committee,* and in Dr. Breslich's article in the *Third Yearbook of the National Council,* any teacher can find adequate suggestions as to where the functional relation can be emphasized in geometry. However, in spite of all that has been said and written about it, we still find teachers who need to be convinced that the functional relation should be noticed in teaching. Some of them do not know what it means.

The word *function* implies *variable.* A function is a variable.

A variable quantity is one whose value changes. For example, the length of the day (daylight), the price of cotton, the radii of different circles, intervals of time, the annual rainfall at a given place, etc.

We say that a function of a variable is a quantity which depends on the variable for its value. This is true but it is not the best expression for the definition of a function. In more precise language, if two variables are so related that for any value assigned to one of them one or more definite values are determined for the other, the second variable is said to be a function of the first. The interdependence between the variables is the most noticeable relation expressed by the definition of function. We may find illustrations of the functional relation everywhere. The circumference of a circle depends on the length of its radius; the distance traversed by a body falling from rest toward the center of the earth depends on the length of time it has been falling; the annual receipts at any postoffice depend on the population of the town and surrounding country served by the postoffice, the enrollment of the public schools of any city depends on the population of the city; the number of overcoats sold annually in any locality depends on the number of degrees registered by the thermometers of that locality; the number of bushels of wheat per acre depends on the number of inches of rainfall, the number of insect enemies, and other things; the price of cotton depends also on the rainfall, the temperature, the boll weevil, and other things.

The mathematical expression of the first two functions mentioned in the preceding paragraph furnish excellent illustrations of the meaning of constant, variable, and function. Consider the relation $C = 2\pi R$. 2 and π are constants, R is a variable and, as written, C is a function of R. The expression $R = C/2\pi$ shows R as a function of C. Such interdependence exists in all functions. Again, $s = gt^2/2$ expresses the law of falling bodies (under constant acceleration in a vacuum) where t is the number of seconds elapsed in the time of flight and s the number of feet traversed in t seconds. Here g and 2 are constants, t is a variable and s a function of t. Some functional relations are thus simply expressed by a mathematical equation. Some are not so simply expressed; for example, the last illustration above, depending on several variables simultaneously. Some have never been expressed in mathematical form other than being pictured by statistical graphs, for they involve too many variable conditions of such irregularity. They are functional relations, nevertheless. Upon man's ability to discover and express mathemati-

cally many functional relations of nature and geometry has hinged the development of much of the science of modern engineering and many other branches of applied science. Because of the ever present functional relations (some expressible and some not) it is imperative that we consider the subject in connection with a course in high school mathematics, though the formal definition of function is not necessarily considered and the name probably not mentioned.

A class was studying "Shop Problems in Mathematics." Some problems were like this: If a cylindrical measure 10 inches deep holds a bushel, what is its diameter? What is the diameter of a half-bushel measure of the same depth? Of a peck measure of the same depth? After answering the first question by solving $V = \pi r^2 h$ for r when $V = 2150.4$ and $h = 10$, the class of boys was practically a unit in declaring that the diameter of the half-bushel measure would be half the diameter of the bushel measure of the same depth, and the diameter of the peck measure one-fourth the diameter of the bushel. The teacher showed them in what way V changed as r changed and, conversely, how r changed as V changed and corrected their erroneous notions. This was a simple consideration of how r depends on V, or in what sense r is a function of V.

Another type of problem was this: If two sewer pipes, each one foot in diameter, converge, how large must the single pipe be to carry away the sewage without overflow when the pipes are running full? Reversing the idea in a similar problem, how many three-quarter-inch house-service pipes will a four-inch water main supply without decreasing the pressure? These involve simple functional relations in geometry; namely, the area of a circle varies directly as the square of the diameter and the relation expressed by the Pythagorean Theorem. Many a teacher in problems like these and many other problems has been emphasizing the functional relation unconsciously, and while doing it some have been fighting the introduction of what they thought was the functional relation into elementary mathematics.

If we are to reach one of the important objectives of mathematics teaching, the development of independent and resourceful thinkers, we lose a very great help to that end when we slight the consideration of the functional relation. There is no doubt

of its far reaching influence in mathematics and in the practical experience of life. There is no doubt that it can serve the purpose of encouraging independent thought and for such a use alone it is worth emphasizing.

Corollary 3: *In order to induce the best type of mathematical thinking, the teacher must know the subject far beyond what he teaches.*

A teacher whose knowledge of analytic geometry is fresh has complete command of the situation when teaching graphs of algebraic functions. Compare his situation with that of a certain teacher of algebra in a high school some years ago who asked me if the subject of graphs was very important, or should it be omitted, admitting at the same time that she had never studied graphs. She had not studied college mathematics and graphs had been omitted in her high school course in algebra.

If the fundamentals of the calculus are fresh in his mind, no teacher will need to ask, "What is meant by the functional relation?"

A consideration of the two principal non-euclidean geometries (by Lobachevsky and Riemann), with their substitution of other postulates of parallels and the consequent change of theorems, will give a teacher a new and more wholesome viewpoint of the subject of geometry. It will force him to think.

A young man who was ranked very high in mathematical ability had almost completed the work for a Master's degree, majoring in mathematics, when he secured a position as teacher of mathematics in high school. His general attitude, manner of speech, and reactions toward educational methods were such that it seemed as if he might be a failure as a teacher of boys and girls. He had a minimum preparation along the lines of psychology and education. At the end of the first year his superintendent was asked for a confidential statement concerning the young man's work as a teacher. He wrote as follows:

> His only weakness in high school is the problem of discipline. He gets so interested in the development of the subject matter that he does not notice slight disturbances at all. It is a fact that he can get more mathematics across to the students in spite of trivial disturbances from time to time than I have ever been able to do and I have been regarded as a strict disciplinarian.

Scholium. A study of the history of geometry as an integral

part of the course will give the pupil an appreciation of its value in connection with human progress from the time when the ancient scholars struggled with their empirical formulas of mensuration to the present day when the results of geometry are used by the astronomers to measure millions of light years out into space in the discovery of great new galaxies of stars outside our own stellar universe. Give the pupils an inspired view of what geometry has meant to the world and as future citizens they will never question the right of geometry to a prominent place in the curriculum.

On page 31 of his new book on *Mathematics in Liberal Education* Professor Cajori says:

"The great mathematicians of Greece, Euclid, Archimedes, Apollonius of Perga, Diophantus, were busy enlarging the field of mathematics and perfecting its logic, but made no explicit declaration concerning its value in training the mind. Our first witness, Protagoras, was the head of a school and expressed himself explicitly in favor of the free elections of studies and against compelling pupils to take subjects which they did not like. In his school mathematics was not a prerequisite. On the other hand, Plato and some of his disciples placed extraordinary emphasis upon the mind training value of mathematics. This training value was also stressed by the two great Greek physicians, Hippocrates of Cos, and Galen, and by the orator Isocrates. We find, therefore, that Protagoras opposed mathematics as necessary for mind training, while Plato, Aristotle, Xenocrates, Isocrates, Geminus, Ptolemy, Hippocrates of Cos, Galen, Nicomachus, Theon of Smyrna, Plutarch and Proclus stressed its great educational value. The count is 12 to 1."

WHAT THE STATE TEACHERS COLLEGE MEANS TO BEMIDJI FINANCIALLY

A Junior High School Arithmetic Project

BY MABEL FRANCES RICE

State Teachers College

Bemidji, Minnesota

Interesting units of work occupying several days or weeks of school time, growing out of the work in History, Geography, Health, Community Life, or English, have not been unusual in the last few years. Entire projects in Arithmetic have been more rare. The children of the seventh and eighth grades of the Junior High School department of the Bemidji State Teachers College completed a six weeks' project that was unique in that arithmetic was the subject that formed the nucleus of the study.

The project was, "What the State Teachers College Means to Bemidji Financially." The problem arose in a social science class which was studying the western movement in United States history. Bemidji is a small city in northern Minnesota in the region built up by the lumber interests but in which that industry is now on the wane. The class traced the evolution of industry. They saw the frontier trading post and the thriving lumbering center, and they made a survey of the present situation in which the enterprising little city is casting about for a new means of livelihood. The industries following upon the heels of lumbering were listed in order of importance with the summer resort industry and the tourist trade at the top of the list. The question at once arose: Does the income brought to Bemidji annually through the State Teachers College warrant placing it among the leading industries of the city? How much does the college bring to Bemidji annually? Nobody knew. Neither the college administration nor the business men around town would hazard a guess. Here was an unsolved problem yet one capable of being solved, the solution of which would be of interest to the whole community. It was a situation to whet the

arithmetical appetite of a seventh and eighth grade class. They set to work with enthusiasm.

Only the funds which remained in the city were to be considered. In a class exercise the various sources of income were listed. They included expenditures of faculty members, other employees of the college, out-of-town students, running expenses of the institution, and local expenditures for materials and labor on the new building under construction.

The next task was to prepare questionnaires for faculty members, students, janitors, engineers, and workmen, asking for an estimate of all possible expenditures. Special questionnaires were prepared for the college accountant, and for the contractor and the superintendent of construction on the new training school building. These questionnaires were worked out in the English classes. Those for the students, the accountant, and the contractor follow. Those for the faculty and the workmen were similar in the essential details and are omitted here.

QUESTIONNAIRE FOR STUDENTS

To the student:

The seventh and eighth grades of the Junior High School department are interested in learning what the State Teachers College brings to Bemidji financially. The seventh grade is to work up the statistics for the student body. May we have your cooperation? Please answer the following questions to the best of your ability. If you are not sure of some amounts will you please approximate them? No item is too small to be listed.

1. Do you live in Bemidji?.....
2. Do you board at the Dormitory?.....
3. Do you room at the Dormitory?.....
4. Do you room in a private home?.....
5. How much do you spend for the following items?
   a. Board per week $.....
   b. Room per week $.....
   c. Clothing per month bought in Bemidji $.....
   d. Laundry per month $.....
6. If you own or rent a house or an apartment:
   a. House rent per month $.....
   b. Taxes per year $.....
   c. Electricity, gas, telephone, fuel, water, etc.....
   d. Groceries, milk, ice, etc.....
7. Did you buy a car in Bemidji?
   a. What make?.....
   b. Garage rent per month $.....

c. Repairs per month $.....
d. Gasoline and oil per month $.....

8. What do you spend for each of the following per year?
   a. Doctor $.....
   b. Dentist $.....
   c. Hospital $.....
9. How much do you spend for each of the following items per month?
   a. Motion picture shows $.....
   b. Other amusements and recreations? $.....
10. List any other expenditures not given above:
   a. .....$.....
   b. .....$.....
11. If there were no college in Bemidji, would you have gone to college in another city?

QUESTIONNAIRE FOR COLLEGE OFFICE

Will you please fill in the following items for us?

1. Total amount paid to faculty for a year $.....
2. Total amount paid to all other employees for a year $.....
3. Fuel for one year $.....
4. Electricity bill for one year $.....
5. Water bill for one year $.....
6. Gas bill for one year $.....
7. Supplies bought in Bemidji (not including food for Sanford Hall) $.....
8. Miscellaneous expenditures $.....
   a. Telephone $.....
   b. Drayage $.....
   c. Printing $.....
   d. .....

QUESTIONNAIRE FOR CONTRACTOR

To the contractor:

The eighth grade is interested in finding out the source of the supplies which you are using in the construction of the new building. May we have your assistance? Will you please estimate any figures which may be difficult to give exactly?

1. How many thousand bricks will you use?.....
   a. Approximate price per M.....
   b. Where obtained.....
   c. From what company.....
2. How much steel is to be used?.....
   a. Price $.....
   b. Where obtained?.....
   c. What company?.....
3. How many thousand feet of lumber?.....
   a. Price per M.....

b. Where obtained.....
c. What company.....
d. How much western lumber do you use.....
4. How many barrels of cement will you use?.....
a. Price per barrel?.....
b. Where obtained?.....
c. Company .....
5. How many temporary posts do you use?.....
a. Price .....
b. Where obtained.....
c. Company .....
6. How many yards of gravel will you use?.....
a. Price per yard.....
b. Where obtained.....
c. Cost for hauling.....
7. Face brick $.....
8. Roofing $.....
9. Payroll $.....
10. Number of men employed.....

The whole project was explained by the children themselves before the general assembly of faculty and students. These three minute assembly talks had been prepared as oral composition in the English classes. The questionnaires were distributed among the faculty, other employees, and students to be filled in, after which they were redistributed among the pupils of the seventh and eighth grades.

Each child was held responsible for a minimum number of questionnaires but was permitted to work at his own rate of speed. The handling and organization of the work presented many arithmetical problems. Attention is called to the fact that on the questionnaires some items are given by the week, others by the month, and still others by the year. To estimate the total yearly expenditure on each one of six hundred questionnaires entailed a vast amount of calculation. Many careless mistakes were made in the beginning. An eighth grade boy found that one student was paying $2,150 a month for board instead of $21.50. A few such errors made every child alert to ridiculous mistakes and forced him to make sensible estimates. By the end of the six weeks period the pupils were making relatively few mistakes. All work was finally checked on an adding machine.

After the pupils had surveyed the individual papers and had estimated the amount expended, they approached the task of

organizing results and totaling the figures obtained from all questionnaires. They ascertained not only the total expenditures of each group but the average expenditures as well.

"What is the total amount which the college has brought to the city since 1919, the year the college was founded?" The class was curious to learn that amount. To answer that question, the pupils collected old college catalogs and from these they estimated the total number of out-of-town students that had attended the school and the length of time each one had spent there. The average current expenditure for the present students was used to approximate roughly the amount which the student body had brought to the city in each one of the past years. The accountant was able to state quite accurately the amount expended annually for running expenses for all years.

The whole class worked feverishly on the last step of the problem which was to find the grand total that the Teachers College had brought to the city financially.

At their own suggestion the pupils made a chart showing some of the interesting facts gleaned from the study. For instance:

1. Each student spends an average of $9.51 a week in Bemidji.
2. Faculty members spend an average of $154.08 a month in the city.
3. Nine members of the faculty own cars purchased in Bemidji.

The children even proposed a hypothetical problem: If there had been no college in Bemidji, how many of the home town students would have had an opportunity to get a college education? How many would have gone away to school? Since thirty percent of the present student body stated that they would have gone elsewhere to school, the class proceeded to estimate the amount of money that the college was keeping in the city. (These figures were not added to the grand total since they were a mere guess.) The remaining seventy percent of home town students who had stated that they would not have gone away to college was made the subject of an interesting little article by each child, written in the English class, showing that other than financial advantages were rendered to the city by the State Teachers College.

There were a number of worth while results of the project.

1. Since about ninety-five percent of all the arithmetic involved was in the four fundamentals and simple percentage, the children received a vast amount of drill in those important processes.

2. The pupils learned the importance of making sensible estimates.

3. The importance of accuracy made itself evident at every turn.

4. The community as well as the pupils derived an interesting fund of information and a greater appreciation of the state institution within its gates.

The findings of the pupils were published in the local papers. The whole project created much interest and comment among the business men and townspeople in general. It brought forth favorable editorials from a number of newspapers. Any school, public or private, could profitably launch a similar project.

Members of the National Council will please note the officers of the organization on the front inside cover page. Many of these officers are changed from time to time, but the members are not aware of these changes.

Note in particular the changes in address of Dr. Vera Sanford, Associate Editor of the TEACHER, and Mr. Edwin W. Schreiber, Secretary-Treasurer of the Council.

A LAYMAN LOOKS AT MATHEMATICS[1]

BY JOHN R. P. FRENCH

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One's philosophy about anything is necessarily based upon one's experiences with that particular thing. I can't remember my mathematical experiences in the grammar school—evidently they made no profound impression. At the tender age of ten I was consigned to the——Latin School, and I remember my mathematical experiences there with some vividness. They were painful. I was perpetually " conditioned " in arithmetic. I felt myself absolutely and finally and completely incompetent in arithmetic. In the language of childhood I " hated " arithmetic—and in any language I certainly dreaded and feared it.

Then I went to the——Latin School. The dear old gentleman who tried to teach me algebra there—how well I remember him! He was patient with me beyond my deserts, I think. He always told me that I *could* if I *would*. But I didn't believe him. I failed my elementary algebra for college entrance—failed it badly.

When I came to geometry, it was different, somehow. I not only got through it—I felt in some small degree at least the fascination of it. I believe we even had a little solid geometry,—because I remember the word parallelopipedon. Perhaps (I throw this out as a very late afterthought, of course) perhaps I had a dim feeling that geometry was something *real*, whereas algebra had always been just a bad dream, a nightmare, a totally meaningless bugaboo. Anyway, I passed my geometry.

I elected no mathematics in college. But in connection with college, a rather curious experience came in. It ought to have been, as I look back upon it, an illuminating experience; but to the best of my recollection it was not. My entrance condition in algebra dangled over my head until the last possible moment—fall of my junior year. It had to be passed off then, or I

[1] Read at the annual meeting of the Association of Teachers of Mathematics in New England in Boston, December 8, 1928.

wouldn't be a junior. I passed it off, after about ten days, or perhaps two weeks, of solitary concentrated study, with no help but my old dog-eared textbook. When I had learned the result of the examination, I thanked "whatever Powers there be," and fervently renounced mathematics for life.

After college, I began teaching science, at first, in a Massachusetts high school. I am old enough to be willing to confess with only a passing wave of shame that the arithmetic of chemistry and the problem-solving of physics were pretty much beyond me, in those first two years. But I bluffed along cheerfully enough somehow, using I don't remember what pitiful stratagems to conceal my weakness. I do remember, however, that about this time I began to enjoy the process of solving problems in physics. (As most of you doubtless know, the problems in any ordinary physics text require for their solution a certain type of ingenuity which most people would call mathematical ingenuity, because it is undoubtedly similar to the ingenuity needed for the solution of most arithmetical and algebraic problems. I say "most people" because I have a notion myself that what is really involved here is first of all *reading* skill—common or garden ability to read the English language with comprehension—and second a trick of seizing upon the main point which is not at all exclusively mathematical. I wonder how many of you mathematics teachers would agree with me in this?—What do you call it? (How about the good old New England term "horse sense"?)

But that is a digression. I remark that I began to enjoy the solution of physics problems, because I think there is a certain significance in the fact. The particular implication is, of course, that I had never enjoyed anything of the kind before, and never supposed I *could* enjoy anything of the kind. Let's just note that point, for future reference.

In my next job, in a boys' boarding school, I was asked to teach arithmetic and elementary algebra, along with certain science courses. I was twenty-four years old—and I had not yet learned the multiplication table. Will you let me ask some questions here, to satisfy my own curiosity? How many of you believe that you knew the multiplication table, so that it was permanently automatic, by the time you had finished the Sixth Grade?—the Eighth Grade?—the High School?—College?—

Well, I learned it, of course. Also I learned—or re-learned, if you will, the processes of elementary algebra. I found no difficulty in learning them, and not much in teaching them. I mention that also to convey a point, to which I shall later on revert.

One more mathematical experience I feel called upon to record. Not many years ago, I was employed for a summer to tutor a fourteen-year-old-boy in Latin. In the family was a sister who had finished her first year of college, but had a condition in advanced algebra to make up. I had never "taken" advanced algebra—but I taught her advanced algebra—or at any rate we learned advanced algebra together, successfully. She got 96 percent in her make-up examination. She had really an excellent mathematical mind—but she was a timid child, lacking in self-confidence. And some freshman instructor had persuaded her that she was hopeless so far as mathematics was concerned. She has since become a competent assistant in astronomical research work. I suspect I don't need to elaborate the moral of that tale.

Now—why all this shameless autobiography? Well, because it seems to me the most genuine approach I can make to whatever I have to say about the teaching of mathematics. For it is, I shall now confess, about the human side of mathematics teaching that I wish to speak, however briefly and feebly. Perhaps I ought to say, the two human sides—because there is the teacher, and here are the scholars. Both, I hope, are human.

Now let me revert, if you will, to certain points in my painful history. Let's take, first, arithmetic at the——Latin School. The point of interest is not that I hated it and feared it and flunked it—but *why?*

I'm going to suggest two possibilities. And, by way of elimination, let me say that I have no recollection whatever of the teacher or teachers who labored with me there. So it cannot have been a personal matter. I think it was a professional matter, entirely. And the two possibilities are—first, that I was too young to learn the particular kind of arithmetic that was prescribed in the course of study; and second, that this was not really the case, but that the method of presentation used sufficed to keep the whole matter beyond my comprehension.

Let's talk first about this matter of maturity. Long ago, I

hope, we all got over the idea that calendar age was a sufficient basis for school grading. Lately, we have talked a lot more about mental age. So far, so good—but I venture to believe there is a further and equally necessary step. I believe *school subject age* needs a lot more careful diagnosis than it has ever yet received. (We test it in the outcome—but how often do we test it in advance of administering the prescription?)

I believe that in mathematics, more than in any other subject except formal grammar, this peculiar aspect of mental maturity, this matter of ripeness or non-ripeness for approach to the particular subject, is fundamental.

Let me give you one or two illustrations drawn from my own recent experience:

1. A boy of fifteen whose generalized ability to do school work is about ten points below norm. He has been in the school five years, and has been moved along from grade to grade regularly in spite of inability to do successfully the standard work of any grade. He is a faithful worker, is now in Grade IX, and has within this year developed a surprising ability to do the Algebra of this grade. He gets 100 percent success in all mechanical parts of the subject and has more difficulty with problems, but tackles them courageously. In short, he has quite unexpectedly "matured" in mathematics although he is still unable to do other work of comparable grade.

2. A girl of fourteen of unquestionably better than average school ability a year ago found beginners' Algebra an insurmountable obstacle. She was a highly conscientious worker but she was allowed to drop Algebra, substituting extra hours in the studio. She has started Algebra again this year and is succeeding perfectly.

3. A boy of sixteen with "very superior" generalized school ability. In 1926–27 worked hard but unsuccessfully at algebra for half of the school year. In 1927–28 he completed successfully a year of plane geometry. This year he is doing algebra at a rate which his teacher estimates as twenty times faster than he was capable of two years ago.

I could quote other instances, if time allowed. I've been watching this thing for ten years—ever since the wisest old language teacher I ever knew remarked to me one day "Formal Grammar, in any language, is an analytical science. It is stupid to try to teach an analytical science to any boy less than fourteen years old." That teacher, by the way, had never so much as heard of "educational measurements"—but his intuitive diagnosis of individual pupil or subject weaknesses was a marvel to behold. He was school tutor in a boys' boarding school; and "got results," in the college-entrance-examination sense.

But, you will say, all this is matter for the administrator, not for the classroom teacher. To be sure it is; but who will do anything about it if the teachers don't? In this matter it is they who are the experts, they and nobody else who can gather the convincing data and compel the blind administrators to "sit up and take notice."

Now what about the other point, the matter of presentation? Well, we do those things better in these days, of course. But I shall venture to thrust upon you here certain laws of the Medes and Persians which I undertake (annually!) to lay down for my own teachers, in any and all subjects.

"You must *satisfy* the pupil," I say, "with respect to the following questions:"

1. *What is it all about? I.e.*, no school subject can safely be assumed to mean *anything* to a beginning pupil. All too often, it remains forever a meaningless task.
2. *Why should I want to do it?* Is it something I can *use*, now or later? If not, does it "train my mind"? How? Or, is it just another one of those things I have to do in order to get into college?

In order to obey the first law it is necessary, of course, for the teacher to perceive some true and genuine relationship between the subject in question and human life as a whole—preferably, too, human life as lived by the particular group of children in front of him. I am sadly conscious of the fact that for years I took it for granted, in teaching classes, that the subject I was introducing was as important to the children as it was to me. The teacher who has discovered that, after all, it is only the children who are important, and not the subject, has taken the first fundamental step toward good teaching.

In order to obey the second law, it is necessary for the teacher first to be well-informed, and second to be rigidly and courageously honest.

But let's be a little more concrete. What shall we say, and what shall we not say, to these young minds, eager or apathetic as the case may be? It looks easy for arithmetic, harder for algebra, and not quite so hard for geometry. But there are pitfalls.

For instance, fifteen years ago Charles W. Eliot, addressing an audience of Massachusetts teachers, said this: "Of all the subjects taught in modern schools arithmetic, beyond the simplest

applications of the simplest rules, is the least practical because its reasoning is demonstrative; whereas the reasoning we have to depend on in the daily conduct of life is almost all probable, and not demonstrative reasoning." And in another sentence: "Arithmetical reasoning is absolutely peculiar, and has very little application in common life."

There, sure enough, is a pitfall for the unwary. And his remark applies, does it not, just as surely to algebra, and to Q. E. D. geometry? No—let's not lie to our children, even in self-defense.

What we are getting into here is, you see, the matter of "transfer" of training. Dr. Eliot was doubtless familiar with the memory-training experiments performed by William James upon himself and others in 1890 or thereabouts. James was a bit surprised to find out, as you probably know, that quite a deal of practice in memorizing Milton's "Paradise Lost" not only failed to have a beneficial effect upon his ability to memorize Victor Hugo, but actually had the opposite effect! (The first recorded instance, so far as I know, of "negative transfer.") But Dr. Eliot could not know, in 1913, that experiments performed during the past five or six years indicate pretty conclusively that the transfer of training in arithmetic reasoning to other kinds of reasoning is about 10 per cent. Just in case this may be news to some of you, let me be specific. A group of children is tested in arithmetic reasoning (ordinary textbook problems) and also in general reasoning. They are then trained in arithmetic reasoning until their improvement is approximately 100 per cent. Then they are tested again in general reasoning power. Their improvement in this turns out to average about 10 per cent.

So apparently we must, however regretfully, find some other basis for justifying the problem work in arithmetic—and probably also in algebra. Can we find one—or several? Why, I should hope so. But if not—and here I fall back on my privilege as a layman, and not a mathematician—if not, then the sooner we drop arithmetic problems and algebra problems out of the curriculum, the better. For I am interested in children, and in their education; and unless any given portion of school subject matter educates children, or somehow conduces to their education, I am all for wiping it off the slate and start-

ing fresh. (Which reminds me, curiously enough, now that I think of it, of certain remarks of more or less similar tenor made in this very room last year by Mr. Smith, who certainly did know something about mathematics.)

But before I go hunting with you for reasons why we should teach what we do teach, let me revert for a moment to another point in my mathematical autobiography. I actually got up the courage, on one occasion, to ask the dear old gentleman who tried so patiently to teach me algebra in the——Latin School why it was that I should study algebra. He made it all quite clear to me, somewhat in the following fashion: "Your mind," he said, "is like a muscle. The harder you exercise it, the bigger and stronger it will grow. Algebra is one of the best exercises ever invented." I believed him of course—but I just simply didn't choose that form of exercise, I suppose. I played chess, with eagerness and assiduity—even carrying a pocket chess-board around with me. But it never occurred to me to think of that as exercise. I did it for fun.

Now the mental discipline theory, after the unmerciful pummeling it has received for half a generation at the hands of the educational psychologists, ought to be dead. But I dare anybody to prove it is dead. I thought myself, ten years ago, that it certainly was dead. But now I don't know. A thing with such enormous vitality in it—even if it was invented by Prussian school men of the deepest classical dye in order to justify the retention of Latin in the curriculum of every German schoolboy—a thing which appears to have more lives than the proverbial cat deserves some modicum of respect, after all. And even if we refuse, as of course we must, to set the value upon it that its authors or my old master in the——Latin School set—still it may be worth some thinking.

I venture to think myself (and I say this with all due diffidence, as a layman in many things besides mathematics) I venture to think myself that perhaps its vitality is due to its unconscious translation, in most people's minds, into a *moral* discipline theory. I hazard the guess that what bothers the psychologists is precisely its moral implications—its moral fringes, so to speak—*which they can't measure.*

Common sense and common observation tell us, after all, that something worth while *does* happen to the boy who uses his

mental powers, vigorously and consistently. He may not be learning anything that "transfers"—but may he not be forming a habit? And might it not be called "the habit of successful mental accomplishment"? And if so, will anyone deny that it is a valuable habit? I suspect our psychologists would want to call it an "attitude of conation," or something of that sort—but whatever its name, it is, I believe, the reality, the core of truth, that makes mental discipline so hard to down.

And if we can so teach our arithmetic and our algebra and our geometry as to foster and to entrench this habit, then we have at least one honest leg to stand on, haven't we? Of course there will appear the conscientious objector who argues "Why not get the same result with content material of more demonstrable life-value, or culture-value?" I shall not attempt to argue that point with him here; but if I were a mathematics teacher I should be tempted to ask him, slyly, what other school subject with a developed teaching technique offers equal opportunity for the *demonstrable* cultivation of this particular moral habit. For in mathematics, as in no other subject, success is a 100-per-cent certainty. Either the answer is right, or it is wrong. And what mathematics teacher does not know the satisfaction that lights the face of even the dullest pupil when finally, after struggle, the answer *is* right?

But, of course, in order that this habit of successful mental accomplishment may indeed be formed, your mathematics must be taught for success—not for failure. Again I revert to my lamentable autobiography. From my own early training in mathematics I acquired, beyond a shadow of doubt, the habit of *un*successful mental accomplishment—the habit of failure. That this was unnecessary and in fact inexcusable seems to me to be fairly well indicated by my own subsequent experiences in mathematics. No—the trouble was not mathematics, nor me—it was the way the two were brought together.

What should my teachers have done? First, I think, they should have found out whether or not I was mentally mature—ripe—for the particular stuff they were going to feed me. If they found I was not, they should have refused pointblank to accept any responsibility for the damage they were about to do to my character. If they found I was, they should have insisted above all things that I *succeed*, and get the full satisfaction

of success, in the first task they gave me. Habits—mental or physical or moral—are formed by repeated actions. We *tend to repeat* those actions which have been accompanied by a feeling of satisfaction. *We tend not to repeat,* but rather to avoid those actions which have been accompanied by a feeling of dissatisfaction. Plato knew this, 2000 years ago; and the modern educational psychologist calls it the "Law of Satisfaction." Law or no law—it is a fact of human nature which cannot be disregarded with impunity.

If we had time, I should like to discuss with you at some length what I believe was the fundamental reason why these good and devoted men taught me to be a failure, in mathematics. It is a pet theory of mine, and I am always anxious to find somebody who will prove to me that it won't hold water. If I thought it were a phenomenon of past history only, I should not venture even to mention it. But I am constrained to believe that it persists, in some degree at least, in the subconscious minds of certain otherwise estimable present-day teachers, and since I believe it to be infinitely damaging and dangerous, both to themselves and to the children, I shall give it a brief airing here.

I believe my teachers in the old days—and to some extent my parents—labored under a philosophical delusion that was the direct product of an unholy alliance between the mental discipline theory and that most vicious tenet of the Calvinistic theology, the doctrine of innate depravity. (You young folks may laugh, if you like—but bear with me while I go on and explain this to your elders who, like myself, were born and bred in the New England of the latter nineteenth century!) Rooted and grounded deep within us of the old New England stock, I affirm, is the conviction that children are born bad. For over a hundred and fifty years of our life as a people on this soil the most popular and widely used school text was the Bay Psalm Book, with Catechism. If you have never seen it, and want to purge your own soul, by all means procure a copy. "In Adams's fall we sinned all," it says—and much more of the same. Only by unlimited doses of "correction and reproof" can we hope to escape the consequences of this original sin—and only then if by some unexplained whim of an inscrutable divine Providence we become one of "the elect."

Thus it was that our forefathers grimly explained to them-

selves the naughtinesses of children, their unwillingness to learn, their natural resistance to all goodness. Generation after generation they explained it so—and is it any wonder that their descendants find it hard to forget?

Now bring to one of those descendants, who is perhaps a teacher of arithmetic in the——Latin School, the fresh, bright, new-minted, convincing doctrine of "mental discipline." We improve our morals most, this man knows in his bones, by the most arduous moral labors, by self-denials, by "mortification of the flesh," by "correction and reproof." What easier corollary, now, than this—that the job the pupil most dislikes and dreads is not only the best training for his mind, but also the best medicine for his soul? What else, given this triumphant "scientific" vindication of an inborn theological prejudice, can anybody believe?

Observe how neatly and completely the dawning of this new idea relieves the teacher of all responsibility for considering the pupil's capacities or tastes. The boy dislikes arithmetic, doesn't understand it, dreads it. "Aha!" says the teacher, "He is a lazy little rascal. He has no moral backbone! I'll give him a still harder task, and make a man of him!"

Does it still persist, this idea, anywhere that you know of? If so, if you know any existing victims of it, beg them first to read Plato—who knew better, 2000 years ago—and then to begin studying children.

There is a discipline, of course, which inheres in hard tasks successfully performed. It is, I have already ventured to suggest, a moral discipline. It is based upon a fact of human nature, which psychologists call the "Law of Satisfaction." It needs no props, either from a badly-discredited theory of "mental discipline" or from a theological dogma which most humane people have long since discarded, intellectually—even if they were born and bred in New England.

That was a long digression, and again I apologize for it. Is there anything else, worth mentioning, that the honest mathematics teacher can and should utilize in presenting himself and his subject? Of course, being mathematics teachers, you know a thousand things. But I shall venture to mention one more, because I think it fails to get the attention and respect it deserves, rather generally. It is the inherent fascination of mathe-

matics. Call it a game or a sport, if you wish. It is there, and it is undeniably and eternally true. The whole realm of mathematics is a series of triumphs of human ingenuity; and as such it inherently possesses the same sort of fascination for growing minds that machinery possesses, that chess possesses, that bridge possesses, that "inside" football and baseball possess. It gets somewhere, with astounding and relentless precision. It is a marvelous human achievement—and, being human ourselves, we are flattered by it, and want to share in it. And, by virtue of every increment in our skill at it, we can "win." We can get the right answer. We can solve the problem. We can increase our self-respect!

So I would say, to teachers of mathematics, by all means do not fail to capitalize this intellectual and emotional fascination which inheres in your subject. If you don't feel it yourselves, you ought not to be teaching mathematics. And if you do feel it, you can make growing minds feel it—yes, even the duffers.

I mean that last seriously—in fact, most seriously. I have already quoted you one case, and I could quote others. I believe implicitly that there exists no normal child who cannot, if taken at the right time and in the right way, get some of this satisfaction and self-respect out of mathematics. Of course it is not good logic to say that if mathematics is good for some children it is good for all children, but I am convinced that it is on the whole good psychology. Only, of course, the dose must be suited to the patient.

And here I come to my final point, which is, *consider the patient*. I use that word patient merely as a convenience, because I am going to use the words "diagnosis" and "prescription." The medical profession is a long way ahead of us, here—so I am borrowing their familiar phraseology.

You don't need to be told, I am sure, that in any good ordinary eighth grade the best pupil in arithmetic reasoning measures up to be 7.5 times as good as the poorest pupil: the best in addition 15 times as good as the poorest; the best in subtraction 8.5 times as good as the poorest; the best in multiplication 17 times as good as the poorest—and so on. (Those particular figures come, by the way, from a study reported in Starch's "Educational Psychology," published in 1927.) The enormous range of individual differences, not only in generalized school

ability but also in specialized subject ability, no longer staggers us. But we must, of course, do something about it. I have no doubt many of you are using the diagnostic pre-tests and the unit-type assignments devised or at any rate described by Miller [2] of Wisconsin and Morrison of Chicago. If you are, you know that within any type of school organization and even with a fairly large class it is possible for you *first to find out definitely what your individual pupil needs* and then to feed him a prescription which will meet his individual need, no more and no less. The Miller concentric-circle type of unit assignment enables you to hand out the same identical nimeographed sheet to each member of your class, and yet know that within it is material easy enough for your dullest pupil, and material hard enough for your brightest pupil. And you must know in advance, of course, just what it is fair to expect from your dullest pupil, and be satisfied and *let him be satisfied* with that much definite achievement; and on the other hand not be satisfied at all with the same achievement from your brightest pupil.

And you will find—I say it with the utmost confidence because over and over again it has been found true in my own school—you will find that your brightest pupil will not himself be satisfied with anything less than complete mastery of the toughest material that you put in the outer circle—and often not with that. He will come to you and ask, of his own accord, for "some more hard ones."

And, after all, why not? You have said to him—you have said to the class—"This stuff out here is pretty hard. I doubt if many of you can do it." A challenge: a subtly implied flattery—what more does any boy or girl with a real mind want? Try it, if you haven't already.

Individual diagnosis and individual prescription, within a strong and coherent social framework. That is what good teaching is coming to mean. And immediately it comes to mean that, it is almost, but not quite, automatically humanized. For if the sting of injustice and the corruption of too-easy tasks are taken out of the relationship between teacher and taught, the sure foundations of harmonious progress are pretty safely laid.

Almost—but note quite. For if you think too much of mathematics, and not enough of Johnny Jones, then not all the clever

[2] Miller, Harry, *Creative Learning and Teaching*, Scribners.

devices of modern pedagogy will rescue you from well-deserved failure, as a teacher. No matter how modern we are or may ever become, no skill or science can ever take the place of that rational devotion, that sympathetic insight, that warm human interest and appreciation which has always been the mark of the true teacher, and always will be. Mark Hopkins on the end of his log can never be replaced by the near-sighted man with the measuring stick. Years ago George Herbert Palmer said and wrote that the first qualification of a teacher was vicariousness—which in the language of an equally great teacher with a pithy vocabulary translates into "Get inside the other fellow's skin, and look out through his eye-holes."

The following quotations from Professor Cajori's new book, *Mathematics in Liberal Education*, pages 38 and 45, respectively, will be of interest to our readers.

"Strange to say, we were not able to find a writer during the Roman period, or that of the Middle Ages, who declared himself against the mind-training value of mathematics. The writers, twenty in all, whom we quoted as explicitly setting forth their convictions of the educational value of this science, are found not only in Rome and central Europe, but also in Spain, Egypt, Turkestan and India."

"It is worthy of observation that during the fifteenth and sixteenth centuries many of those who recommend the study of mathematics on account of its ability to strengthen the mind were teachers of mathematics."

"We have quoted also the observations and views of teachers like Roger Ascham and Joseph Scaliger, primarily interested in other fields, and of religious reformers and literary men not in the teaching profession. Of all the witnesses, twenty-three favor mathematics, while four—Joseph Scaliger, Montaigne, Fracastorius and the Prince of Miranda—are opposed to mathematical training. Ascham favors mathematics in a limited amount, but warns us against overmuch and exclusive study of pure mathematics."

INDIVIDUAL WORK IN ALGEBRA

BY MARTHA C. COOKE

Thomas Jefferson Junior High School, Cleveland, Ohio

The value and efficiency of individual work as compared with group work is much discussed and is far from settled. Many educators feel that individual work in all studies is as good or better than class work. Others feel that at least in "tool subjects," where discussion is of minor importance and the object is only to master a technique or acquire a skill, the individual work plan can be used more profitably than class work.

The following report of individual work done in ninth grade algebra classes is offered only as additional testimony in the discussion and not with the intent to be in any way conclusive. Mathematics classes were selected for the work, since mathematics is usually granted to be primarily a tool subject. Similar work was carried on in a superior group and in an average group. The classes were taught by different teachers. Some slight variations in method were necessitated by the different texts used by the two classes, but on the whole the work was comparable. The 9Bx (superior) group used as text the Thorndike Junior High School Mathematics, Book III. The 9By (average) class used Nyberg, First Course in Algebra.

The Plan

The classes were told that members were to be permitted to work individually and as rapidly or as slowly as desired, that occasional group discussions would be held and that all other help would be given in individual conferences with the instructors. The work was divided into blocks. For each block, page assignments in the text and leading questions were given out in mimeographed form. Answers to problems were also mimeographed and given to pupils or posted on the bulletin board.

The pupils were instructed to work conscientiously on a block of work, checking their answers by means of their answer sheets and to raise questions, if they needed to do so, but to work things

out for themselves as far as possible. When a pupil felt that he understood the work in a block, he was to ask the teacher for a test on the material. If he passed the test successfully he was to go on to the next block. If he failed, he was to go back and study further on that block and a second test was given.

In the 9Bx section the work covered the fundamental processes in algebra and the solution of simple equations and problems. In the 9By section the work covered positive and negative numbers, the fundamental processes with monomials, the addition of polynomials, and the multiplication of polynomials by monomials. The material for the two classes was entirely different and no comparisons were made between the two.

Duration of the Work

No definite time limit was set at first, but the classes were told that when the majority of the pupils had finished all the blocks, a date would be set for a review test covering the work of all the blocks. The actual time used in the 9By section was five weeks; in the 9Bx section about seven weeks. The X section, however, included considerably more material in their work.

Class Discussions

Due to the fact that the pupils worked at different rates of speed (there was at times a difference of as much as four blocks between the slowest and fastest pupils) group discussions except on very general topics were unprofitable after the first week or two of the plan. Discussions in small groups of five or six pupils, however, frequently took place and seemed to serve the purpose better than discussions in which the whole class tried to join.

Individual Help

The amount of individual help needed by pupils varied greatly. Some pupils moved along with only occasional questions, seeming to get all the help they needed from the mimeographed hints and questions given out for each block. Other pupils needed help and encouragement at every turn and never felt sure of themselves.

Tests of Mastery

Short tests were given when each individual completed a block of work, and a final review test on all material covered was given at the end of the work.

Records

Large charts containing the names of all the members of the class, and the blocks of work, were posted. When any pupil completed the work and passed the test for a given block, he made a check mark opposite his name and beneath the block number. Thus any member of the class could see his position relative to all others in the class at any time.

Provision for Those Who Finished their Assignment Early

Pupils who finished the assignment before the others were permitted to study advanced work or to do any recreational work of a mathematical character. In the 9By section, for instance, many of the pupils made attractive posters showing magic squares made with algebraic quantities.

Reaction from the Class

At the end of the work the opinion of the pupils of both classes was solicited. Questionnaires concerning the work were given them, and since their replies were anonymous, a free expression of opinion was the result. A summary of their opinions follows:

Questions	*Replies*	
1. Is individual method of instruction easier or harder for you than the class method?	9Bx	9By
	7	10 Easier
	9	6 Harder
	1	5 Same
2. Were the tests given sufficient to check on mastery of the topics?	11	10 Sufficient
	6	11 Insufficient
3. Did you feel the need of more class discussion?	12	8 Yes
	5	13 No
4. On the whole did you like the individual method?	11	17 Yes
	6	4 No

5. Would you like (*a*) to continue this method entirely, (*b*) to discontinue it entirely, (*c*) to continue it with modifications? ...	6	4 Continue
	4	4 Discontinue
	7	13 Modify
6. Did you find that the posted record sheet stimulated you to work?	11	19 Yes
	4	2 No
7. Do you feel that you learned better or not so well under this method?		4 Better
		3 Not so well
		14 Same

8. List some advantages you see in this method:

Advantages

a) Pupil can work at his own speed.
b) Pupil becomes self-reliant.
c) Pupil learns things better (9Bx only).
d) Pupil learns to think for himself.
e) Pupil feels the satisfaction of accomplishment (9Bx).
f) Pupil can adjust the amount of work to his own wishes and to the amount of his other homework.
g) Pupils waste less time.
h) Pupils can check on their own work by means of the answer sheets (9Bx only).
i) There is not so much review.
j) No advantages (one pupil).
k) Chart a stimulus for harder work.

9. List the disadvantages you see in this method:

Disadvantages

a) Pupils try to keep up with others and thus work too fast (9Bx only).
b) It's hard for pupil to work things out for himself.
c) Pupil can't learn as well without more class discussion.
d) Hard for slower pupils.
e) Pupil learns too easily and things "don't stay with him" (9Bx only).
f) Pupil has to judge his own ability to take a test.
g) Pupils waste time (too much freedom).
h) Pupils can cheat on answers perhaps.
i) Everything (one pupil).
j) Teacher has to make several explanations of same work to individuals, instead of one to the class.
k) End test not hard enough.
l) Small tests not complete.
m) There is more work to do.
n) Some children don't do all the work assigned.
o) Not as interesting as group method.

Conclusions

In summing up the work, it will be well to give the instructors' opinions of the method as worked out in their classes. First of all in most cases the children liked the method and it was gratifying to see their pride and pleasure when they had worked out some rather difficult point entirely for themselves. It was, of course, also gratifying to the pupils. Some pupils developed a useful technique for attacking a problem. Others went down before each new difficulty and never acquired confidence or skill. On the whole the brighter children in both classes were the ones who liked the method and profited from it, although ideally it should be equally good for the slower ones. On the other hand, the necessity for mastering one topic before beginning another gave the children the feeling of being abreast of their work and eliminated discouragement to a great extent.

Many pupils, being told that they might work at their own rate of speed, did little or nothing at first, and later, in an effort to catch up with others in the class, attempted to cover a week's work in a single night. This resulted in superficial work in these cases.

In the 9By class some of the pupils did not play the game and cheated on the small tests which were given at different times to different pupils. This showed up of course in the final test. In the 9Bx class, no cheating was apparent in the results of the tests.

The material which required only skill was mastered efficiently by pupils under the individual method. Material which required greater understanding (*e.g.*, the use of formulas and the methods of deriving new formulas from given ones, in the 9Bx class) was not learned or understood as well as with group teaching.

It would seem then that the individual method as used in these classes would be useful in mastering material which is purely tool material, but that material which requires or admits of greater understanding (not just skill) is less effectually learned with this method than with group teaching.

ABSURDITIES DUE TO DIVISION BY ZERO

An Historical Note

BY FLORIAN CAJORI

University of California

The uncanny "demonstration" that 1 is equal to 2, or 3 is equal to 9, as a consequence of a division by zero, has found its way into publications on mathematical recreations and modern text books. If $a = b = 1$, then $a^2 - b^2 = a - b$, $(a + b)(a - b) = a - b$, and $a + b = 1$, or $2 = 1$. Somewhat more generally, if $a = b = m$, where m is any number, real or imaginary, then $a + b = 1 = 2m$. If $m = \frac{1}{2}$, every thing is normal, but if $m = \sqrt{3}$, then $\sqrt{3}$ becomes equal to a rational number; if $m = 3 + 3i$, then $6 = 1$ and at the same time $6 = 0$.

Who was the first "demonstrator" of these fallacies? They are not found in books on mathematical recreations before the close of the nineteenth century. They are not found in the first two editions of W. W. R. Ball's *Mathematical Recreations*, both of which appeared in 1892. But in the third edition, 1896, it is shown that $1 = 2$, and the remark is added in a footnote that this fallacy is "well known." Nevertheless, Ball appears to be the first to give it as a "recreation" exercise.

Before tracing the origin of this trick, we desire to point out that the earliest writer to define the operation of division in ordinary algebra and to draw rigidly the inference from that definition that division by zero is impossible in that algebra, was Martin Ohm,[1] the brother of Georg Ohm of "Ohm's Law" in physics. Martin Ohm gave the proof in 1828. But long before that date, Bishop Berkeley in his *Analyst* (1734) had asked the question of mathematicians in his "Query 16": "Whether certain maxims do not pass current among analysts which are shocking to good sense? And whether the common assumption, that a finite quantity divided by nothing is infinite, be not of this number?" A rigorous argument demanding the exclusion of division by zero is found also in a book of 1872, by Robert Grass-

[1] Martin Ohm, *Versuch eines vollkommen consequenten Systems der Mathematik*, Vol. I, 2d Ed., Berlin, 1828, p. 112.

mann,[2] but in elementary school algebras it did not appear until about the year 1881. None of these references answers the question as to the ingenious spirit who first evolved the paradox that $1 = 2$, or $a = b$, where a and b are not the same.

Recently the desired information stared us in the face as we were reading, for another purpose, Bernard Bolzano's *Paradoxien des Unendlichen*, first published in 1851. Bolzano (1781–1848) was professor of the philosophy of religion at Prague, but was also a mathematician of marked power, who anticipated Weierstrass and some other mathematicians in a few of their discoveries. His mathematical researches were long overlooked by mathematicians, and some of his results have only recently been published. Bolzano was familiar with Ohm's book and states that Ohm was the first to call the attention of the mathematical public to the difficulties surrounding the concept of zero. After explaining that zero cannot be taken as a divisor, Bolzano proceeds as follows.[3]

"If a and b are a pair of distinct magnitudes: then the two following identities hold:

$$\begin{array}{l} a - b = a - b \\ b - a = b - a \\ \hline a - a = b - b \\ a(1 - 1) = b(1 - 1). \end{array}$$

If now it is permissible to divide the two numbers of an equation by a factor which is equal to zero, we obtain the absurd result $a = b$, whatever a and b may be. But it is generally known that in larger computations one comes all too easily upon wrong results, when one removes a common factor from both sides of an equation without first ascertaining that it is not zero."

Here we have the earliest presentation of the fallacy resulting from division by zero. The mode of procedure is not quite the same as Ball's but in both we have the absurdity arrived at through the illegitimate operation of division by zero. Accordingly, not only did Bolzano anticipate Weierstrass in discovering

[2] R. Grassmann, Formenlehre *oder Mathematik; Viertes Buch, Zahlenlehre oder Arithmetik*, Stettin, 1872, p. 5, 20. I am indebted for this reference to Mr. O. Schmiedel.

[3] B. Bolzano, *Paradoxien des Unendlichen*, von Dr. Fr. Pihonsky, 2d edition, Berlin, 1889, p. 59.

a continuous curve which has no derivatives, but he anticipated Ball and other modern writers in one of the most widely known arithmetical pastimes of our day. However, it is true that with Bolzano the absurdity in question was not a "recreation," but a serious "paradox." For a long time, apparently, deriving an "absurdity" was not considered a "recreation."

It is notorious that in preliminary investigations mathematicians frequently do not take account of all the fine points necessary in exact thought. The precautions are taken later at more leisure moments. At the Congress of Arts and Science at St. Louis in 1904, Maxime Bôcher expressed himself as follows: "There is what may perhaps be called the method of optimism, which leads us either willfully or instinctively to shut our eyes to the possibility of evil. Thus the optimist who treats a problem in algebra or analytical geometry will say: 'I know that I have no right to divide by zero; but there are so many other values which the expression by which I am dividing might have that I will assume that the Evil One has not thrown a zero in my denominator this time.' "

NEWS NOTES

The General Science Quarterly, published for twelve years by Professor W. D. Whitman of Salem Normal School, Salem, Massachusetts has become the official organ of the National Association for Research in Science Teaching. It will, hereafter, be published under the title SCIENCE EDUCATION and will devote its pages to the teaching of science in all grades from the kindergarten to college. The first issue under the new name appeared in June 1929. A sample copy of this issue may be obtained by writing to Science Education, Salem, Massachusetts. The journal is being edited by a committee of the National Association, composed of

Walter D. Whitman, Salem Normal School.

Earl R. Glenn, Montclair State Teachers College.

Charles J. Pieper, New York University, Chairman.

That the fourth Yearbook of the National Council is being appreciated abroad is evidenced by the following letter to the Council from Professor Kornesh, Secretary of State for Education in Hungary.

"I have just received your Yearbook and I hasten to express my sincere thanks therefor. The chapter concerning Hungarian conditions I have found to be thoroughly correct and comprehensive."

The eleventh annual meeting of the National Council of Teachers of Mathematics will be held in Atlantic City, N. J., February 21st and 22d, 1930. The topic for the meeting will be "Geometry in the Curriculum of the Junior and Senior High School." Further particulars concerning the meeting will be given later.

The following program will be given at the Mathematics Section of the Wisconsin State Teachers Association at Milwaukee, Wisconsin, in the Athletic Club, 410 Broadway, on Thursday afternoon, November 7, 1929:

12:30 P.M. Luncheon, $1.00. Notify officers and come.
2:00 P.M. Address—Mathematics and The Eternal Verities—Dr. L. C. Karpinski, University of Michigan.
2:45 P.M. Discussion.
3:00 P.M. Business Intermission.
3:15 P.M. Address—The Use of the New Type Examination in the Teaching of Mathematics—Miss Mary A. Potter, Racine High School.
3:45 P.M. Discussion.
4:00 P.M. Adjournment.

Officers

I. N. Warner, State Teachers College, Platteville, Wis.—Chairman.
Miss Marie Johnson, 232 Broadway, Eau Claire, Wis.—Secretary.
Miss Ethel Carter, Appleton High School—Program Committee.

The Department of Mathematics of Central High School in Oklahoma City now has 100% membership in the National Council of Teachers of Mathematics.

Miss Elizabeth Rogers of the Charles E. Coon High School in Wilson, N. C., writes: "I inclose $2.00 for which you will renew my subscription to *The Mathematics Teacher*. It is a good magazine—much better than those put out by teachers of other subjects I think. I would hate to be without it." This is similar to a great many letters which the *Teacher* receives frequently. These letters help to encourage those who are doing their best to keep the magazine on a high plane.

The following program was given at the Mathematics Conference of the Michigan Schoolmaster's Club on Friday, April 26, 1929.

Chairman: Lucille MacKenzie, Mt. Clemens.
Secretary: Warren R. Good, University High School.

Friday Morning

1. Introductory Discussion of the Work of the Conference Committee on Definition of Units of Subject Matter Led by the Chairman.
2. Summaries of the Recommendations of Certain Agencies Regarding Secondary School Courses in Mathematics.
   a. The Report of the National Committee. Helen K. Mackintosh, Supervisor of Mathematics, Grand Rapids.
   b. The College Entrance Examination Board. Richard McDade, Lincoln Consolidated School, Ypsilanti.
   c. The North Central Association. Ruth Utley, Northeastern High School, Detroit.
   d. The New York State Board of Regents. J. P. Everett, Western State Teachers' College.
   e. City and State Courses of Study. Nellie Loss, Flint.
3. Suggestions on Technique to Be Used by Conference Committee. Mildred Crawford, Roosevelt High School, Ypsilanti. C. L. Thiel, Detroit Public Schools.

12:15 P.M.
Luncheon

Friday Afternoon

4. The Relation Between the Research Department of City Schools and Instruction in Mathematics. L. D. Lundberg, Department of Research and Statistics, Flint.
5. Two Representative Problems and Their Historic Setting. Edwin W. Schreiber, University High School, Ann Arbor.
6. What Is a Progressive Program in the Teaching of Mathematics? John R. Clark, Lincoln School of Teachers College, Columbia University.

The Bay Section of the Mathematics Teachers Association of Cali-

"Progressive, c'est moi!"
J. R. C.

fornia held its spring meeting on April 20, 1929, in Berkeley. Miss Jean Tuttle is president of the Section. The Committee Chairmen were:

General Reception

H. B. Lathrop, San Jose High School, San Jose.

Program for Luncheon

Mr. Mueller, Roosevelt High School, Oakland.

Decoration

Miss Giddings, Frick Jr. High School, Oakland.

Exhibit

Miss Smith, Horace Mann Jr. High School, S. F.

Miss Hoeffer, Richmond High School, Richmond.

The program of the meeting follows:

Research in Mathematics

Factorization of Large Numbers: Dr. Lehmer, University of California.

Research Problems: Dr. Rutledge, Director of Research and Guidance, Oakland Public Schools.

Reorganization of Geometry: Miss Allen, University High School, Oakland.

Modern Trends in Teaching Mathematics: United States—Miss Eaton, Oakland High School, Oakland; Germany—Dr. Noble, University of California; England and France—Mr. Yulich, Our Past President.

Business Session.

Luncheon and Exhibit: Faculty Women's Club.

Round Table Discussion.

Tests and Measurements: Leader Mr. McCarty, Lowell High School, S. F. Discussion Miss Hennessy, S. F. H. S. of Commerce, Miss Hesse, University High School.

Modern Methods: Leader Mr. Albrecht, Fremont High School. Discussion Miss O'Neil, Lakeview Jr. H. School, Mr. Clawson, Technical High School.

The Mathematics Department of the Eusley High School of Birmingham, Alabama, with 12 teachers now has a membership of 100% in the National Council of Teachers of Mathematics.

YOUR FRIENDS, THE PUBLISHERS

There has long since ceased to be any doubt about the constructive services of the textbook publishers to public school development. The day of questionable methods in the marketing of textbooks has also for the most part slipped into the limbo of forgotten things. Publishers are still the most potent force in curriculum development in America in spite of a generally awakened interest in this activity on the part of educators. The friendly attitude that uniformly exists between buyer and seller in the textbook business is a praiseworthy condition.

But publishers are sometimes thoughtlessly imposed upon by teachers, and a word needs to be said in this connection. The bane of the business is the terrific drain in the giving of free samples of textbooks. Perhaps publishers were originally at fault in encouraging this practice, but in fairness to them teachers ought now aid in checking the practice. Textbooks could be sold at lower figures if this item of selling expense were materially reduced. And the teacher does hold much of the power of improvement in her own hands.

Publishers are glad to have teachers manifest enough interest in some new publication to write for a copy, but why should the teacher expect to get that copy without paying for it. Frequently there is no possibility whatever of its being considered for general use, but it is sent for merely because the teacher thinks it might be of value to her. In no other line of business would that same teacher expect to get a full-sized package of a product that interested her, although samples are sometimes offered. One cannot write for a tire for the automobile and expect it to be sent gratis on the theory that if satisfactory it

may be adopted for all four wheels. Even a box of face powder cannot be obtained with the maker's compliments in the hope that by thus submitting it he may gain an exclusive five-year contract. Why should the publisher's wares be regarded so differently? If the teacher wants a copy of a new text, the equitable thing to do is to send along the money to pay for it. Or, because administrative heads of school systems are generally expected to be in touch with the newest materials, she can probably obtain a copy from the superintendent's office for such inspection as she wishes to make; or she can ask to have a representative call. Publishers will never escape the necessity of supplying samples to school executives or teacher committees which are definitely charged with the duty of specific text adoptions, but they ought not to be subjected to indiscriminate requests.

The self-respect which makes us unwilling to step into a store and ask a merchant to give us an article off his shelves ought to govern us in our relation with textbook publishers. Let us be just as alert and responsive to publisher's announcements as ever, but let us play the game in a broad-minded way by paying for what we want. Free book samples are not a proper perquisite of the office of teaching.—From *Ohio Schools*, June, 1929.

let us play the game in a broad-minded way by paying for what we want. Free book samples are not a proper perquisite of the office of teaching.—From *Ohio Schools*, June, 1929.

The Secretary of the National Council of Teachers of Mathematics has prepared a handsome engraved certificate of affiliation for all clubs and associations of teachers of mathematics who affiliate with the council. Officers of clubs who are interested should address Secretary E. W. Schreiber, Western Illinois State Teachers College, Macomb, Ill.

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